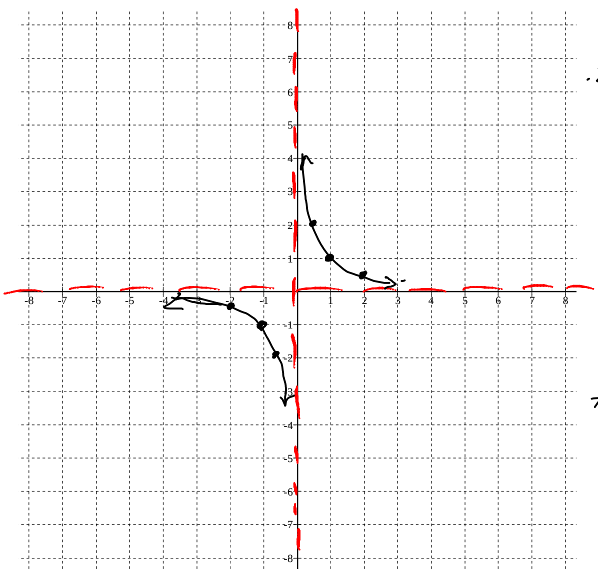
Unit 9: Rational Functions
9.1 Exploring Rational Functions Using Transformations

Rational function: A function of the form $f(x) = \frac{p(x)}{q(x)}$ where p + q are polynomial expressions and $q(x) \neq 0$

* in PREC11 we graphed the simplest rational $y = \frac{1}{x}$ (reciprocal function)



$\frac{1}{x} \neq 0$ ∴ horizontal asymptote $y = 0$

non-permissible value $x \neq 0$ ∴ vertical asymptote $x = 0$

Domain: $\{x \mid x \neq 0, x \in \mathbb{R}\}$

Range: $\{y \mid y \neq 0, y \in \mathbb{R}\}$

* Key points $(\frac{1}{2}, 2)$ $(1, 1)$ $(2, \frac{1}{2})$ $(-\frac{1}{2}, -2)$ $(-1, -1)$ $(-2, -\frac{1}{2})$

* at vertical asymptotes $|y| \to \infty$

We can transform rational functions the same as before, so a function like $y = \frac{3}{x+1} + 4$ means

$3\left(\frac{1}{x+1}\right) + 4$

Transforming the graph of $y = 1/x$:
- vertical expansion × 3 (OR horizontal exp. × 1/3)
- translation 1 left
- " 4 up

$y = \frac{1}{x + \frac{1}{3}} + 4$

Consider the function $\boxed{y = \frac{6}{x}}$

$y = 6/x$ or $y = \frac{1}{x/6}$

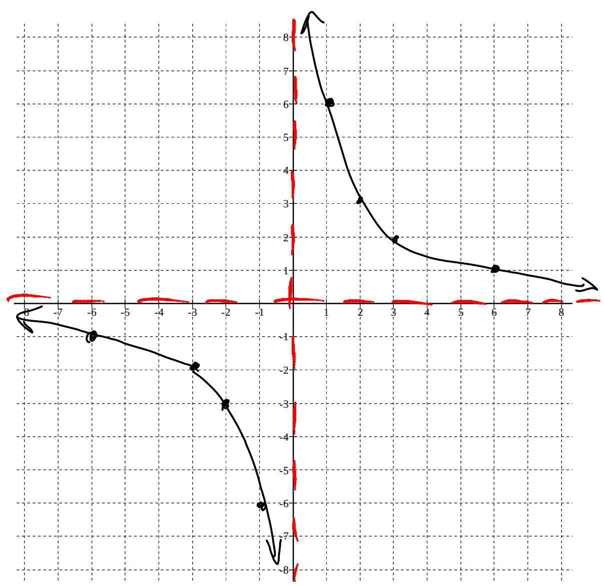
vertical OR horizontal expansion × 6

$y = \frac{1}{x}$		$(x, 6y)$	OR $(6x, y)$
$(\frac{1}{2}, 2)$	→	~~$(\frac{1}{2}, 12)$~~	$(3, 2)$
$(1, 1)$	→	$(1, 6)$	$(6, 1)$
$(2, \frac{1}{2})$	→	$(2, 3)$	~~$(12, \frac{1}{2})$~~

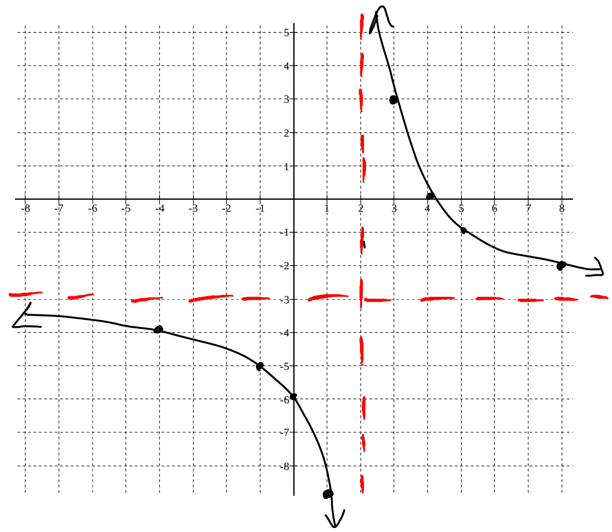
$y = \frac{6}{x}$

$xy = 6$

Two values with product 6



Ex. Graph the function $y = \frac{6}{x-2} - 3$



translations 2 → + 3 ↓

∴ asymptotes

$x = 2 \qquad y = -3$

What if the function is in a different form? For example, the previous function could have been written

$$y = \frac{6}{x-2} - \frac{3(x-2)}{1(x-2)}$$

$$= \frac{6-3x+6}{x-2}$$

$$\boxed{y = \frac{-3x+12}{x-2}}$$

We don't have a specific *vertex* or *standard* form, but we can break them into multiple parts.

Ex. $y = \frac{3x}{x+5}$

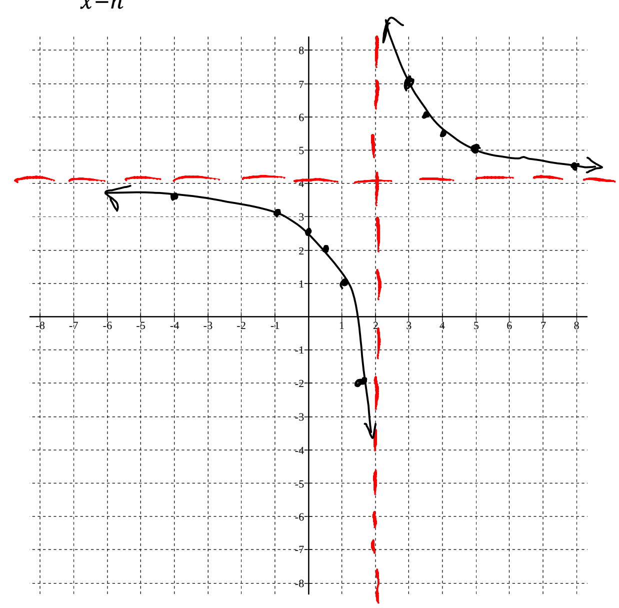
$y = \frac{a}{x-h} + k$ is useful for graphing

$$y = \frac{3x+15-15}{x+5}$$

$$= \frac{3x+15}{x+5} - \frac{15}{x+5} = \frac{3\cancel{(x+5)}}{\cancel{x+5}} - \frac{15}{x+5}$$

$$\therefore\ y = \frac{-15}{x+5} + 3$$

Ex. Graph $f(x) = \frac{\boxed{4x}-5}{x-2}$ by rearranging the function to $y = \frac{a}{x-h} + k$



$$= \frac{4x-5-3+3}{x-2}$$

$$= \frac{4x-8}{x-2} + \frac{3}{x-2}$$

$$= \frac{4\cancel{(x-2)}}{\cancel{x-2}} + \frac{3}{x-2}$$

$$f(x) = \frac{3}{x-2} + 4$$

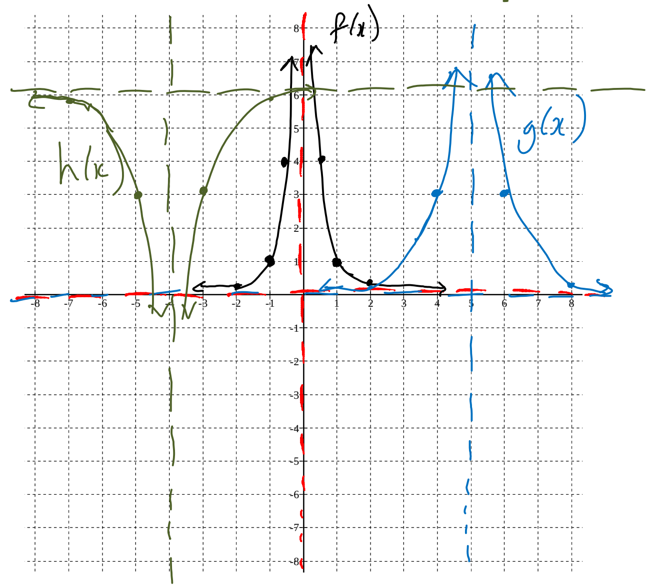
$(\tfrac{1}{2}, 2) \rightarrow (\tfrac{3}{2}, 2)$ or $(\tfrac{1}{2}, 6)$

$(1, 1) \rightarrow (3, 1)$ or $(1, 3)$

$(2, \tfrac{1}{2}) \rightarrow (6, \tfrac{1}{2})$ or $(2, \tfrac{3}{2})$

Ex. Graph and compare the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{3}{x^2-10x+25}$, and $h(x) = 6 - \frac{1}{(x+4)^2}$.

$g(x) = \frac{3}{(x-5)^2}$



$y = 1/x^2 \rightarrow y = 3/x^2$ vert exp × 3

$\rightarrow y = \frac{1}{x^2/3} = \frac{1}{(x/\sqrt{3})^2}$ horiz exp × $\sqrt{3}$

Complete the table.

Characteristic	$f(x) = \frac{1}{x^2}$	$g(x) = \frac{3}{x^2 - 10x + 25}$	$h(x) = 6 - \frac{1}{(x+4)^2}$
Non-permissible value (NPV)	$x \neq 0$	$x \neq 5$	$x \neq -4$
Behavior near NPV	$\lvert y \rvert \rightarrow \infty$	$\lvert y \rvert \rightarrow \infty$	$\lvert y \rvert \rightarrow \infty$
End Behavior	$y \rightarrow 0$	$y \rightarrow 0$	$y \rightarrow 6$
Domain	$x \neq 0$	$x \neq 5$	$x \neq -4$
Range	$y > 0$	$y > 0$	$y < 6$
Equation of vertical asymptote	$x = 0$	$x = 5$	$x = -4$
Equation of horizontal asymptote	$y = 0$	$y = 0$	$y = 6$

Ex. Mobile phone plans:

BASIC -$10 monthly fee and 10¢ per text message or minute of talk time.

SUPER SAVER -$5 monthly fee and 15¢ per text message or minute of talk time.

a) Let x represent the number of minutes, and write rational functions for the average cost per minute for each plan.

$C = \dfrac{\text{total cost}}{\#\text{ minutes}}$

BASIC

$C = \dfrac{10 + .1x}{x}$

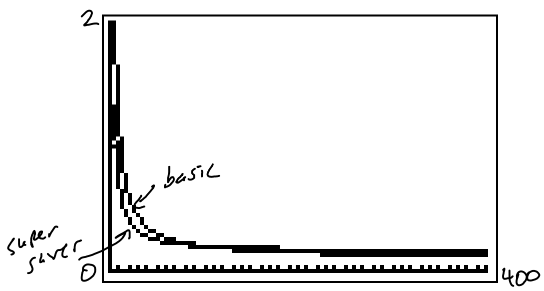
OR $C = \dfrac{10}{x} + 0.1$

SUPER SAVER

$C = \dfrac{5 + .15x}{x}$

OR $C = \dfrac{5}{x} + .15$

b) Graph both functions on your graphing calculator.

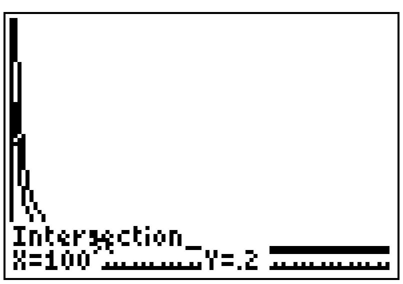


c) What do the graphs show about the average cost per minute for the two plans as time increases? Both have vert. asymp. $x=0$ $\Rightarrow$ cannot have average cost at 0 minutes

As the minutes increase the average cost per minutes approaches 10¢ for the basic & 15¢ for the super saver.

d) Which plan is the better choice?

Find intersection $\Rightarrow$ at $(100, 0.2)$



$\therefore$ Less than 100 minutes – Super Saver better

greater than 100 minutes – Basic better

Practice: pg. 442/ # 1, 3 – 10, 14, 16, 18