Unit 3: Polynomial Functions
3.3 The Factor Theorem

Recall from 3.2:

Remainder Theorem – When a polynomial in x, $P(x)$, is divided by a binomial of the form $x - a$, the remainder is $P(a)$.

A divisor of a dividend that results in no remainder is called a factor.

The Factor Theorem: A polynomial in x, P(x), has a factor of (x−a) if and only if P(a) = 0.

Ex. Which of the following binomials are factors of $P(x) = x^3 - 3x^2 - x + 3$?

a) $x + 2$

$P(-2) = (-2)^3 - 3(-2)^2 - (-2) + 3$
$= -8 - 12 + 2 + 3$
$R = -15$ ✗
NOT a factor

b) $x + 3$

$P(-3) = (-3)^3 - 3(-3)^2 - (-3) + 3$
$= -27 - 27 + 3 + 3$
$R = -48$ ✗
NOT a factor

c) $x - 3$

$P(3) = (3)^3 - 3(3)^2 - (3) + 3$
$= 27 - 27 - 3 + 3$
$R = 0$ ✓
Yes, x−3 is a factor

Now that we know x − 3 is a factor of $x^3 - 3x^2 - x + 3$, how could we proceed to factor the polynomial? [*Hint: if I told you 59 was a factor of 708, how would you proceed factoring 708?]

Divide

$$\begin{array}{r|rrrr} 3 & 1 & -3 & -1 & +3 \\ + & \downarrow & 3 & 0 & -3 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$\underbrace{1 \quad 0 \quad -1}_{x^2 - 1}$

$708 \div 59 = 12$
$708 = 59 \times 12$
$= 2 \times 2 \times 3 \times 59$

$x^3 - 3x^2 - x + 3 = (x-3)(x^2-1)$
$= \boxed{(x-3)(x+1)(x-1)}$

So once we have one factor of a polynomial (which can be verified by the Factor Theorem) we can proceed to factor the entire polynomial, but how do we identify that first factor $x - a$ such that $P(a) = 0$?

Integral Zero Theorem: If x−a is a factor of a polynomial in x, P(x), with integral coefficients, then "a" is a factor of the constant term of P(x).

Ex. Identify the possible integral zeros and the corresponding binomial factors of each polynomial:

a) $x^3 + 2x^2 - 5x - 6$

possible integral zeros $= \pm 1, \pm 2, \pm 3, \pm 6$

possible factors $(x-1)\ (x-2)\ (x-3)\ (x-6)$
$(x+1)\ (x+2)\ (x+3)\ (x+6)$

b) $2x^3 - 5x^2 - 4x + 3$

$\pm 1, \pm 3$

$(x-1)\ (x-3)$
$(x+1)\ (x+3)$

Ex. Given $2x^3 - 5x^2 - 4x + 3$.

a) Factor the polynomial completely.

① possible zeros $\pm 1, \pm 3$

② test with factor theorem

$P(1) = 2-5-4+3 = -4$ ✗

$P(-1) = -2-5+4+3 = 0$ ✓

$\therefore x+1$ is a factor

③ divide

$$\begin{array}{r|rrrr} -1 & 2 & -5 & -4 & 3 \\ & \downarrow & -2 & 7 & -3 \\ \hline & 2 & -7 & 3 & 0 \end{array}$$

④ factor

$2x^2 - 7x + 3$

$\times$ 6, $+$ −7: −1, −6

$\underbrace{2x^2 - x}\underbrace{- 6x + 3}$

$x(2x-1) - 3(2x-1)$

$\boxed{(2x-1)(x-3)(x+1)}$

b) Solve $2x^3 - 5x^2 - 4x + 3 = 0$.

$(2x-1)(x-3)(x+1) = 0$

$2x-1=0$ OR $x-3=0$ OR $x+1=0$

$\boxed{x = \tfrac{1}{2} \qquad x = 3 \qquad x = -1}$

Ex. Fully factor $x^4 - 5x^3 + 2x^2 + 20x - 24$.

$\pm 1, 2, 3, 4, 6, 8, 12, 24$

$P(1) = -6$ ✗

$P(-1) = -36$ ✗

$P(2) = 0$ ✓

$x-2$ is a factor

$$\begin{array}{r|rrrrr} 2 & 1 & -5 & 2 & 20 & -24 \\ & \downarrow & 2 & -6 & -8 & 24 \\ \hline & 1 & -3 & -4 & 12 & 0 \end{array}$$

$x^3 - 3x^2 - 4x + 12$

$\underbrace{x^3 - 3x^2}\ \underbrace{-4x + 12}$

$x^2(x-3) - 4(x-3)$

$(x-3)(x^2-4)$

$(x-3)(x-2)^2(x+2)$

$x^3 - 3x^2 - 4x + 12$

$\pm 1, 2, 3, 4, 6, 12$

$Q(1) = 6$ ✗

$Q(-1) = 12$ ✗

$Q(2) = 0$ ✓

$x-2$ is a factor

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -4 & 12 \\ & \downarrow & 2 & -2 & -12 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$\underbrace{x^2 - x - 6}$

$(x-3)(x+2)$

$\therefore \boxed{(x-2)^2(x+2)(x-3)}$

*What if constant is 0? Ex. $q^3 - 4q$

$q(q^2-4)$

$\boxed{q(q-2)(q+2)}$

Practice: pg. 133/# 1 – 7 esl, 8, 10, 13, 15, 16