Unit 11: Combinatorics
11.3B The Binomial Theorem

Consider the expansion of $(a+b)^n$ for varying whole values of n:

$(a+b)^0 =$ 1

$(a+b)^1 =$ $1a + 1b$

$(a+b)^2 =$ $1a^2 + 2ab + 1b^2$

$(a+b)^3 =$ $1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 =$ $1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$(a+b)^5 =$ $1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

*coefficients reproduce Pascal's Triangle

*exponent of a decreases from n to 0

*exponent of b increases from 0 to n

*degree (exponent sum) of each term is n

*number of terms is $n+1$

Ex. expand $(a+b)^7$

*instead of continuing the pattern completely, continue the pattern of the coefficients (Pascal's Triangle)to row 8 and fill in the variables and exponents (which don't need the row above):

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$\therefore (a+b)^7 = \boxed{a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7}$

Ex. expand $(2x-3)^4$

*use row 5 → $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ and let $a = 2x$, $b = -3$

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$= (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$

$= 16x^4 + 4(8x^3)(-3) + 6(4x^2)(9) + 4(2x)(-27) + 81$

$= \boxed{16x^4 - 96x^3 + 216x^2 - 216x + 81}$

Recall from 11.3A that the numbers in Pascal's Triangle, and thus also the coefficients in these binomial expansions, can also be created using **combinations**.

Ex. Expand $(a^2+2b)^3$

*coefficients from 4 th row in Pascal's Triangle: ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$

1 3 3 1

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$x = a^2$ $y = 2b$

$\therefore (a^2+2b)^3 = (a^2)^3 + 3(a^2)^2(2b) + 3(a^2)(2b)^2 + (2b)^3$

$= \boxed{a^6 + 6a^4b + 12a^2b^2 + 8b^3}$

The Binomial Theorem

The expansion of $(a+b)^n$, n a whole number, is given by:

$(a+b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_{n-1}a^1b^{n-1} + {}_nC_na^0b^n$

t_3

The general term in the expansion (ie. a formula to find any specific term) is:

$\boxed{t_{k+1} = {}_nC_ka^{n-k}b^k}$ *on formula sheet

*note: the value of k is always 1 less than the term we are generating

Ex. Determine the 3rd term in the expansion of $(a+b)^8$

$*n =$ 8

$t_3 = {}_8C_2\, a^{8-2} b^2 = \boxed{28a^6b^2}$

$*k + 1 = 3, \therefore k =$ 2!

Ex. For $(2x - 3y)^{11}$ determine the general term and the 8th term.

General term:

*substitute $n =$ 11, $a =$ 2x & $b =$ -3y

$\boxed{t_{k+1} = {}_{11}C_k(2x)^{11-k}(-3y)^k}$

8th term: ($k + 1 =$ 8, $\therefore k =$ 7)

$t_8 = {}_{11}C_7(2x)^{4}(-3y)^7 = 330(16x^4)(-2187y^7) = \boxed{-11{,}547{,}360x^4y^7}$

Ex. If the expansion of $(x + ay)^7$ includes the term $672x^2y^5$, determine the value of a.

General term:

$t_{k+1} = {}_7C_k\, x^{7-k}(ay)^k = 672x^2y^5$ $\therefore k = 5$

${}_7C_5\, \cancel{x^2}\, a^5 \cancel{y^5} = 672\cancel{x^2}\cancel{y^5}$

$21a^5 = 672$

$a^5 = 32$

$\boxed{a = 2}$

Ex. If the 3rd term of the expansion $(x - 3y)^n$ has coefficient 90, determine the value of n.

General term:

$t_{k+1} = {}_nC_k\, x^{n-k}(-3y)^k$

$t_3 = {}_nC_2\, x^{n-2}(-3)^2y^2$

$= \boxed{9 \cdot {}_nC_2}\, x^{n-2}y^2$

$\therefore 9\, {}_nC_2 = 90$

${}_nC_2 = 10$

$\frac{n!}{(n-2)!\,2!} = 10$

$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 20$

$n^2 - n - 20 = 0$

$(n-5)(n+4) = 0$

$\boxed{n = 5}$, $\cancel{-4}$

Practice: pg. 542/ # 4 – 7, 11, 12, 17 - 20

Worksheet