8.3 Number of Solutions for Systems of Linear Equations

Focus on ...

- explaining why systems of linear equations can have different numbers of solutions
- identifying how many solutions a system of linear equations has
- solving problems involving linear systems with different numbers of solutions

At the Arctic Winter Games, Northern youth share cultural experiences and compete in various events. One of the events is snowshoe racing. Situations involving time and distance, such as a race, can be represented with a system of linear equations. If you solve a linear system involving a race, will you always expect a single solution?

Web Link

To learn more about the Arctic Winter Games, go to www.mhrmath10.ca and follow the links.

Investigate Number of Solutions for Systems of Linear Equations

Materials

- stopwatch or other timer or clock showing seconds
- measuring tape
- grid paper and coloured pencils, or computer with graphing software

Work in groups of four to act out different race scenarios. Each group needs to assign two people to race, one person to measure time, and one person to record data.

1. As a group, design a data table to record start and end times and distances for each racer.

2. Complete several different race scenarios. Each student should move at a constant speed for the entire race. For each race, record the data. Consider the starting line distance zero.

3. Create a distance-time graph to represent each race. You may wish to use a different colour for each line.

4. Write a system of linear equations to represent each race.

5. **Reflect and Respond** Solve each system of linear equations you graphed. How many solutions were there for each linear system? Explain how each solution relates to details of the race.

6. How could you have predicted the number of solutions for each linear system just by knowing the starting points and speeds of the racers?

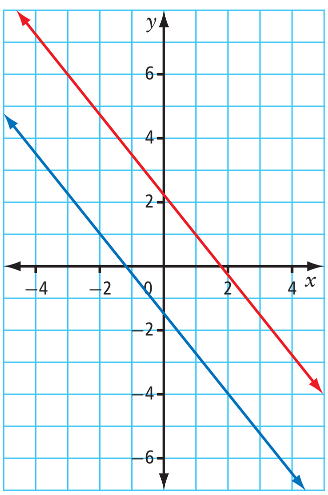
Link the Ideas

When two lines are graphed on the same grid, they do not always have exactly one point of intersection as seen in sections 8.1 and 8.2. Parallel lines do not intersect at all. So, a system of parallel lines has no solution.

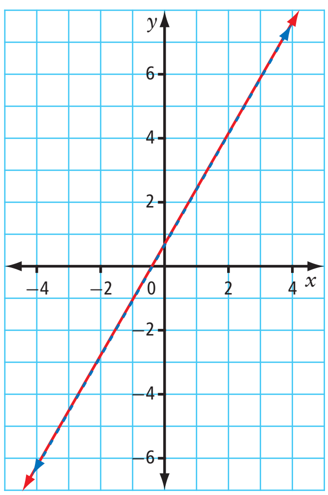
Coincident lines have an infinite number of solutions because the lines are equivalent. They overlap.

coincident lines
- lines that occupy the same position
- in a graph of two coincident lines, any point of either line lies on the other line

Parallel Lines



Coincident Lines



Reducing the equations to lowest terms may help you identify whether the equations are equivalent. If they are equivalent, then they must have an infinite number of solutions.

For the linear system $x - 2y + 5 = 0$ and $3x - 6y + 15 = 0$, the first equation is in lowest terms, but the second equation is not. Dividing each side of the equation $3x - 6y + 15 = 0$ by 3 gives $x - 2y + 5 = 0$, which is equivalent to the first equation. Therefore, the linear system has an infinite number of solutions.

Example 1 Connect the Number of Solutions to the Situation

A particular dog-mushing race is 13 km long. The distances and speeds for several competitors at a certain time during the race are shown in the table of values.

	Current Distance Travelled (km)	Current Speed (km/h)
Competitor A	6.0	24
Competitor B	5.0	32
Competitor C	4.0	24
Competitor D	4.0	24

Assume the racers continue at their current speeds. For each pair of competitors below,

- write a system of linear equations representing their travel from this point forward
- graph each system of linear equations
- identify and interpret the solution to each linear system

a) competitors A and B
b) competitors A and C
c) competitors C and D

Did You Know?

Many breeds of dogs are used for competitive racing. Teams of Canadian Eskimo Dogs have helped people survive in the harsh climate of northern Canada for centuries. Canadian Eskimo Dogs help move people and materials throughout regions that are covered in snow during most of the winter. The Canadian Eskimo Dog is Nunavut's official mammal in honour of its vital role.

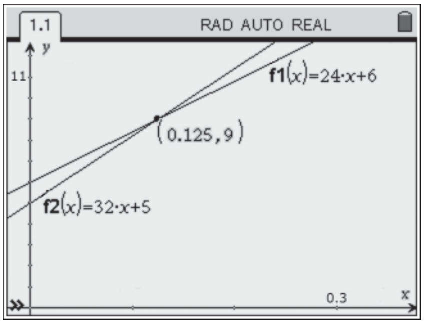
Solution

Let d represent distance from the start of the race. Let t represent time from this point on in the race.

a) Competitor A is travelling at 24 km/h and has travelled 6.0 km. Competitor B is travelling at 32 km/h and has travelled 5.0 km. Their travel can be represented by the following equations:

$d = 24t + 6$

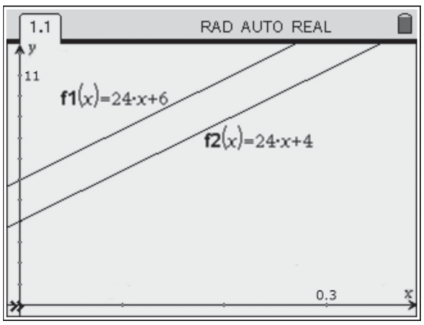
$d = 32t + 5$



The lines intersect at one point. So, the linear system has one solution.

Competitor B will catch up to competitor A after 0.125 h, or 7.5 min, and then proceed past. The point (0.125, 9) is the only point that lies on both lines.

b) Competitors A and C are both travelling at 24 km/h. Competitor A has travelled 6.0 km. Competitor C has travelled 4.0 km. Their travel can be represented by the equations $d = 24t + 6$ and $d = 24t + 4$.

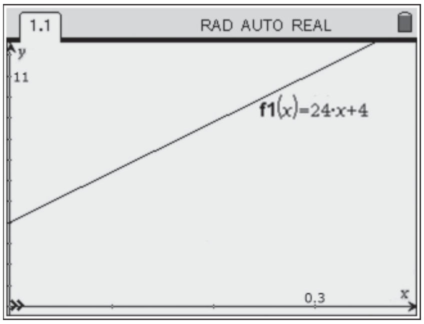


The lines have the same slope but different y-intercepts.

The lines are parallel. They have no points in common, so the linear system has no solutions.

Competitors A and C travel at the same speed, but at different distances from the start line. Competitor C will never catch up to competitor A. There is no point where they are at the same distance at the same time.

c) Competitor C is travelling at 24 km/h and has travelled 4.0 km. Competitor D is also travelling at 24 km/h and has also travelled 4 km. They are currently at the same distance and travelling at the same speed. Their travel can be represented by the equations $d = 24t + 4$ and $d = 24t + 4$.



The lines have the same slope and the same *y*-intercept.

The lines are coincident, so they share all the same points. The linear system has an infinite number of solutions.

Competitors C and D are side by side on the course. They will continue this way because they are travelling at the same speed.

Your Turn

Four vehicles travel on a long, straight stretch of the Trans-Canada Highway. Their current distances and speeds are shown in the table of values.

	Current Distance (km)	Current Speed (km/h)
Car	40	90
Minivan	25	90
Truck	30	110
RV	40	90

For each pair of vehicles, represent the distance-time relationship using a system of linear equations. Suppose the vehicles continue at their current speeds. Identify and interpret the solution to each linear system.

a) the car and the minivan
b) the car and the RV
c) the truck and the RV

Example 2 Predict and Confirm the Number of Solutions

Predict the number of solutions for each system of linear equations. Explain your reasoning, and then confirm each answer by graphing the linear system.

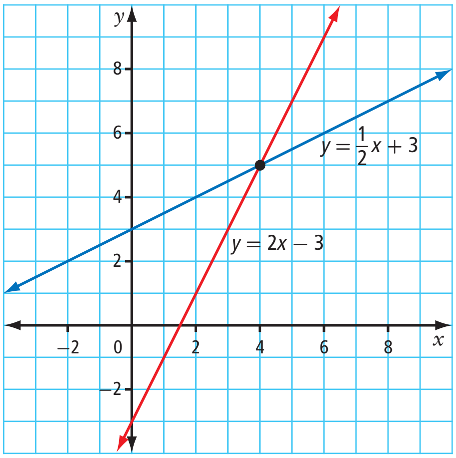
a) $y = 2x - 3$
$y = \frac{1}{2}x + 3$

b) $4x + 10y = 30$
$2x + 5y = 35$

c) $10x - 6y = -12$
$21y = 42 + 35x$

Solution

a) The slope of $y = 2x - 3$ is 2. The slope of $y = \frac{1}{2}x + 3$ is $\frac{1}{2}$.



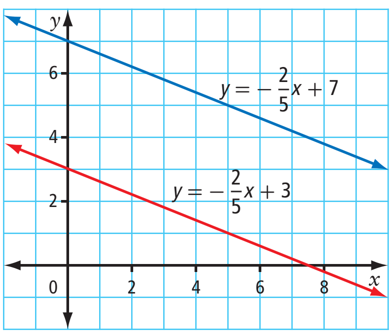
The equations have different slopes. So, the graph will result in two lines that intersect at one point. Therefore, this system has one solution.

b) Rearrange each equation to slope-intercept form by isolating y.

$$\begin{aligned} 4x + 10y &= 30 \\ 4x + 10y - 4x &= 30 - 4x \\ 10y &= -4x + 30 \\ y &= \frac{-2}{5}x + 3 \end{aligned}$$

$$\begin{aligned} 2x + 5y &= 35 \\ 2x + 5y - 2x &= 35 - 2x \\ 5y &= -2x + 35 \\ y &= \frac{-2}{5}x + 7 \end{aligned}$$

Since the lines have the same slope and different y-intercepts, the graph will result in parallel lines. The lines will never intersect. Therefore, this linear system has no solutions.



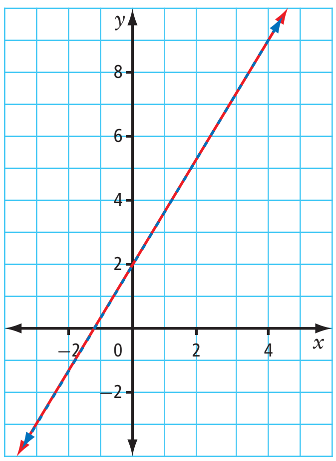
c) For the linear system $10x - 6y = -12$ and $21y = 42 + 35x$, isolate y in each equation to compare the slopes and y-intercepts.

$$10x - 6y = -12$$
$$10x - 6y + 6y + 12 = -12 + 6y + 12$$
$$10x + 12 = 6y$$
$$\frac{5}{3}x + 2 = y$$
$$y = \frac{5}{3}x + 2$$

$$21y = 42 + 35x$$
$$\frac{21y}{21} = \frac{42}{21} + \frac{35x}{21}$$
$$y = 2 + \frac{5}{3}x$$
$$y = \frac{5}{3}x + 2$$

Both equations have a slope of $\frac{5}{3}$ and a y-intercept of 2.

The graph will result in coincident lines. Therefore, this linear system has an infinite number of solutions.



Your Turn

Predict the number of solutions for each system of linear equations. Justify your answers using a graph.

a) $x + 2y = 4$
$y = -\frac{1}{2}x + 4$

b) $6y - 4x = 6$
$y = \frac{2}{3}x + 1$

c) $y = 3x - 1$
$y = 2x - 1$

Example 3 Identify Zero and Infinite Solutions by Comparing Coefficients

Sabrina's teacher gives her the following systems of linear equations and tells her that each system has either no solution or an infinite number of solutions. How can Sabrina determine each answer by inspecting the equations?

a) $2x + 3y = 12$
$2x + 3y = 20$

b) $2x + 3y = 12$
$4x + 6y = 24$

Solution

a) Sabrina notices that the left sides of the equations are identical. So, any ordered pair she substitutes will result in the same value on the left side of each equation. However, the right sides are not equal. There are no ordered pairs that can satisfy both equations, so the lines never intersect. Sabrina concludes that the linear system has no solutions.

How else could you confirm that the lines are parallel?

b) Sabrina notices that the second equation, $4x + 6y = 24$, is not in lowest terms. She divides each term by 2. This results in an equation that is identical to the first equation, $2x + 3y = 12$. Therefore, the equations are equivalent and the graph will be a pair of coincident lines. The linear system has an infinite number of solutions.

$$\frac{4x}{2} + \frac{6y}{2} = \frac{24}{2}$$
$$2x + 3y = 12$$

Your Turn

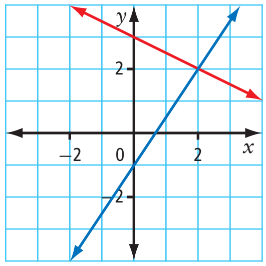
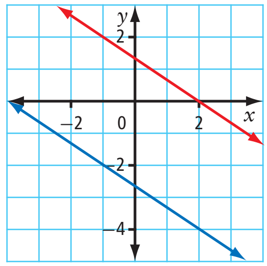
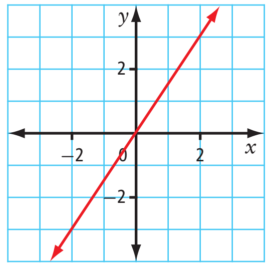
Determine, by inspection, whether each linear system has an infinite number of solutions or no solution. Explain your reasoning.

a) $2x + 10y - 16 = 0$
$x + 5y - 8 = 0$

b) $x + 2y + 4 = 0$
$x + 2y - 6 = 0$

Key Ideas

- A system of linear equations can have one solution, no solution, or an infinite number of solutions.
- Before solving, you can predict the number of solutions for a linear system by comparing the slopes and *y*-intercepts of the equations.

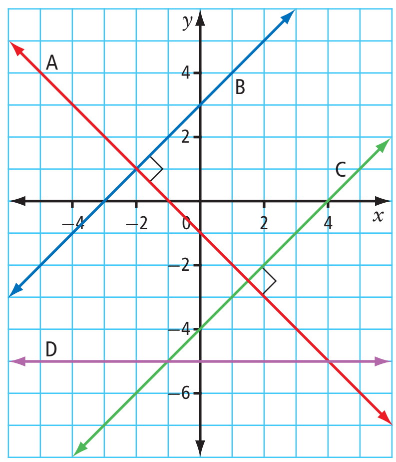
Intersecting Lines	Parallel Lines	Coincident Lines
one solution	no solution	an infinite number of solutions
		
different slopes	same slope	same slope
y-intercepts can be the same or different.	different *y*-intercepts	same *y*-intercept

- For some linear systems, reducing the equations to lowest terms and comparing the coefficients of the *x*-terms, *y*-terms, and constants may help you predict the number of solutions.

Check Your Understanding

Practise

1. Which pair(s) of lines in the graph form a linear system that has
   a) exactly one solution?
   b) no solution?



2. Predict the number of solutions for each system of linear equations. Justify your answers.

a) $y = x + 2$
$y = x + 2$

b) $y = 2x - 4$
$y = x + 1$

c) $y = 3x + 2$
$y = 3x - 5$

3. How many solutions does each linear system have? Justify your answers.

a) $x + 3y = 6$
$y = -\frac{1}{3}x + 6$

b) $3x - y = 12$
$4x - y = 12$

c) $x - 4y = 8$
$x + 4y = 20$

4. Describe the graph and the equations of a linear system that has

a) no solution

b) one solution

c) an infinite number of solutions

5. Describe a strategy for predicting how many solutions each system of linear equations has just by looking at it. Test your strategy by graphing each linear system.

a) $2x + 7y = 28$
$2x + 7y = 15$

b) $x + 2y = 12$
$2x + 4y = 24$

Apply

6. One of the equations in a linear system is $2x - y + 5 = 0$. What might the other equation be if the system has

a) no solutions?

b) one solution?

c) infinitely many solutions?

7. A provincial magazine employs sales people who earn money for every subscription they sell. Several employees are comparing their earnings so far this month.

	Current Earnings ($)	Earnings Per Subscription ($)
Alyssa	472	7.00
Brian	360	8.25
Charlie	360	8.25
Dena	413	8.25

Write a system of equations to represent the earnings for each pair of employees. Identify the solution to each system. Explain how the solution relates to the employees' earnings.

a) Brian and Charlie

b) Alyssa and Brian

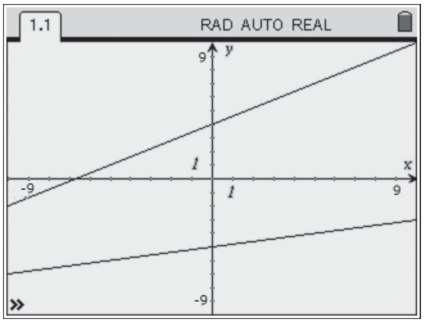
c) Charlie and Dena

8. Sal and Jeff graph the system $y = \frac{5}{8}x + \frac{11}{7}$ and $y = \frac{7}{11}x + \frac{8}{5}$ using technology. They use dimensions of −10 to 10 on the x-axis and −10 to 10 on the y-axis. Sal thinks there is one solution. Jeff claims that there are an infinite number of solutions.

   a) Describe how to predict the number of solutions.

   b) Use technology to recreate their graph. Explain why each person might have come to his conclusion.

9. Stephanie graphs a system of equations using technology. She obtains the following graph and concludes that the system has no solutions.



   a) Explain why Stephanie might have made her conclusion.

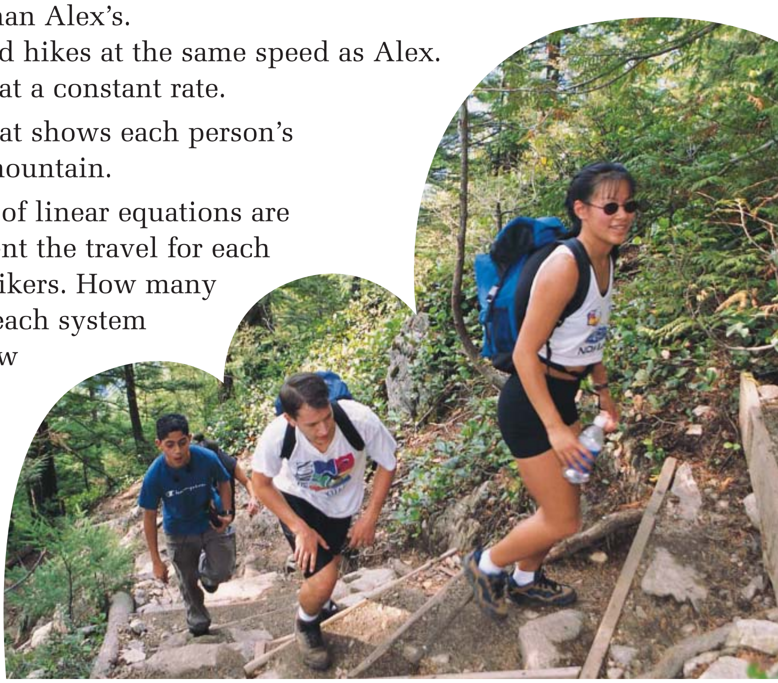
   b) Is Stephanie correct? Explain.

10. Alex, Sandra, Christine, and Jared are hiking the Grouse Grind® near Vancouver, BC.
   - Alex starts first.
   - Sandra and Christine start a short time later. They hike together at a faster speed than Alex's.
   - Jared starts last and hikes at the same speed as Alex.

   Each person travels at a constant rate.

   a) Sketch a graph that shows each person's progress up the mountain.

   b) Suppose systems of linear equations are created to represent the travel for each possible pair of hikers. How many solutions would each system have? Explain how the number of solutions relates to the hike.



Did You Know?

The Grouse Grind® is the most used hiking trail in the Vancouver area. The trail is 2.9 km long and rises 853 m. Over 100 000 people walk or run this steep climb every year. Maintenance workers have built steps up most of the trail to protect against erosion and for the safety of the hikers.

11. Is the following statement true or false? Explain and provide an example to support your answer.

"If a system of linear equations has an infinite number of solutions, then any pair of numbers is a solution to the system."

12. Suppose you are given only the following pieces of information about a system of linear equations. Would you be able to predict the number of solutions to the system? Explain.

a) The slopes of the lines are the same.

b) The y-intercepts of the lines are the same.

c) The x-intercepts are the same, and the y-intercepts are the same.

13. PaperWest has produced 5000 kg of napkins. It continues to manufacture 350 kg of napkins per week. Northern Paper manufactures napkins at a rate of 1400 kg per month and has already produced 28 000 kg. Assume one month has exactly four weeks.

a) Write a system of linear equations to represent the manufacturing of the napkins.

b) Explain how the number of solutions to the system relates to this situation.

Extend

14. For the linear system $2x + 3y = 12$ and $4x + 6y = C$, what value(s) of C will give the system

a) an infinite number of solutions?

b) no solution?

15. Two taxis travel the same route from the airport. One taxi is 6 km from the airport and has a fuel economy of 20 km/L. The other taxi is just leaving the airport and uses 5 L of fuel for every 100 km travelled.

a) Create a system of linear equations relating the distance travelled (y kilometres) to the amount of fuel used (x litres) for each taxi.

b) Explain how the number of solutions to the system relates to the travel of the taxis.

16. Consider the system $y = 56 - 2x$ and $y = 10 + x$.

a) Suppose the domain for the system is restricted to $0 \leq x \leq 8$. How many solutions does the system have? Explain.

b) Suppose there are no restrictions on the domain. How many solutions does the system have?

c) What effects do restrictions on the domain of the equations have when you are predicting the number of solutions? Explain.

17. Consider the system $Wx + 3y = 2W$ and $12x + Wy = 24$.

a) What value(s) of W give the system an infinite number of solutions?

b) How many solutions will the system have for value(s) of W other than those you found in part a)? Justify your answer.

Create Connections

18. Two plants are growing according to the equations $h = 60 + 3t$ and $h = 55 + 2t$. In the equations, h represents height, in centimetres, and t represents time, in weeks. Wendy reasons that one plant is shorter and is growing slower, so it will never catch up to the other plant. Wendy concludes that the system has no solutions. Harriet states that the two lines have different slopes, so there must be an intersection point. Who do you agree with? Create a graph and use it to explain your answer.

19. Can a system of two linear equations have exactly two solutions? Explain your answer, using words and diagrams.

20. Do you think it is always possible to tell how many solutions a system of linear equations has just by looking at the equations? Explain your thinking.

21. Describe a real-life situation that could be modelled by a system of linear equations having each number of solutions. Include examples of possible systems of linear equations.

a) no solution

b) one solution

c) an infinite number of solutions

22. **MINI LAB** Investigate relationships between equations in a linear system and the number of solutions to the system.

Materials

- graphing calculator or computer with graphing software

Step 1 Consider $2x + y = 4$ as equation #1. Create four linear systems (A, B, C, and D) by writing equation #2 according to each of the following instructions.

System A: Multiply (both sides of) equation #1 by a number of your choice.

System B: Add a number of your choice to only the right side of equation #1.

System C: Perform an operation of your choice on both sides of equation #1.

System D: Perform an operation of your choice on only one side of equation #1.

Step 2 Use a graph to analyse the number of solutions for each system of linear equations.

Step 3 What can you conclude about the number of solutions for a system of linear equations if one equation is a multiple of the other? What if two *different* equations in general form have identical coefficients of x-terms and identical coefficients of y-terms?

Step 4 Create an equation #2 so that the linear system has exactly one solution. Experiment with technology to develop several different possibilities. What conclusions can you make about the equations in a linear system that has exactly one solution?