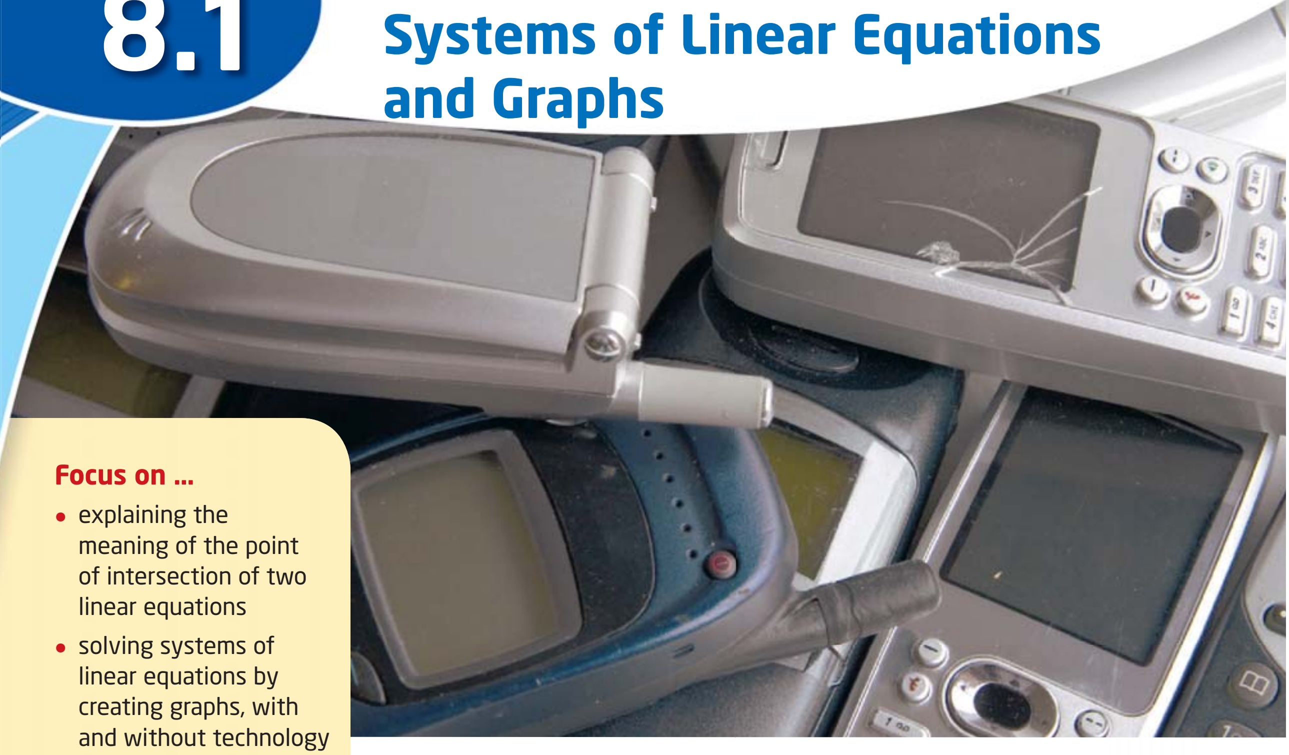
8.1 Systems of Linear Equations and Graphs



Focus on ...

- explaining the meaning of the point of intersection of two linear equations
- solving systems of linear equations by creating graphs, with and without technology
- verifying solutions to systems of linear equations using substitution

The average cellular phone in North America is used for one and a half years and then replaced. Only 5% of discarded cell phones are recycled. That creates large amounts of waste.

Cell phone communication has increased dramatically in recent years. Many people need to decide whether to buy a cell phone, which type of plan is most beneficial to them, and what to do with a cell phone that they no longer need.

Investigate Ways to Represent Linear Systems

Materials

- 0.5-cm grid paper
- ruler
- coloured pencils

How can you compare and analyse cell phone plan options?

- Plan A costs 30¢ per minute.
- Plan B costs $15 one time plus 10¢ per minute.

1. Create tables of values to show the cost of each option for up to 100 min. Use intervals of 10 min.
2. On the same sheet of 0.5-cm grid paper, graph the data from both tables of values.

Remember to include a scale, labels on the axes, and a title on your graph.

3. **Reflect and Respond** From the graph, explain the cost of each plan as the number of minutes increases.

4. What is the significance of the **point of intersection** of the lines? Explain the connection between this point on the graph and the tables of values you created.

5. Which cell phone plan do you think is a better option? Justify your choice.

point of intersection
- a point at which two lines touch or cross

Link the Ideas

Relations can be represented numerically using a table of values. They can be represented graphically, and verified algebraically. A **system of linear equations** is often referred to as a linear system. It can be represented graphically in order to make comparisons or solve problems. The point of intersection of two lines on a graph represents the **solution** to the system of linear equations.

system of linear equations
- two or more linear equations involving common variables

solution (to a system of linear equations)
- a point of intersection of the lines on a graph
- an ordered pair that satisfies both equations
- a pair of values occurring in the tables of values of both equations

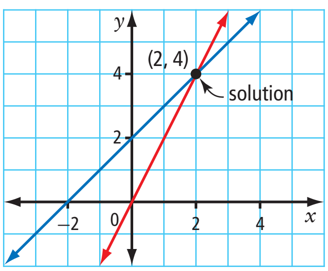
Numerically

x	*y*
0	0
1	2
2	4
3	6
4	8

x	*y*
0	2
1	3
2	4
3	5
4	6

There are an infinite number of pairs of values that could be written in each table. The solution, (2, 4), is the only pair that can be written in *both* tables.

Graphically



There are an infinite number of points on each line. The solution point, (2, 4), is the only point that lies on *both* lines.

Algebraic Verification

$y = 2x$
$4 = 2(2)$
$4 = 4$

$y = x + 2$
$4 = 2 + 2$
$4 = 4$

There are an infinite number of ordered pairs that satisfy each equation. Only one ordered pair, (2, 4), satisfies *both* equations.

Example 1 Represent Systems of Linear Equations

Nadia has saved $16, and her sister Lucia has saved $34. They have just started part-time jobs together. Each day that they work, Nadia adds $5 to her savings, while Lucia adds $2. The girls want to know if they will ever have the same amount of money. If so, what will the amount be and on what day?

Solution

The girls use a linear system to model their savings. They represent it numerically and graphically.

Method 1: Use a Table of Values

The amount of money each girl saves is a function of the number of days worked. They each create a table of values to show how their savings will grow:

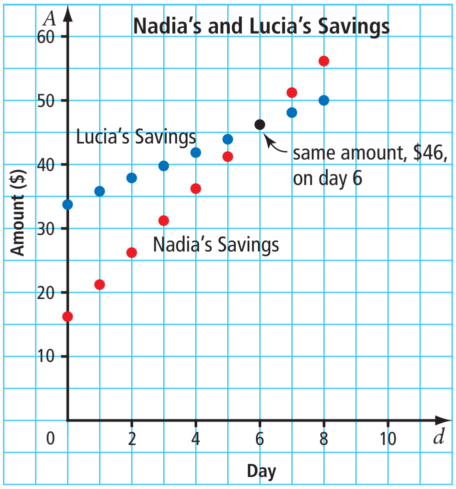
Nadia's Savings	
Day	Amount ($)
0	16
1	21
2	26
3	31
4	36
5	41
6	46
7	51
8	56

Lucia's Savings	
Day	Amount ($)
0	34
1	36
2	38
3	40
4	42
5	44
6	46
7	48
8	50

The tables of values show that both girls will have $46 on day 6. The pair of values, 6 and 46, is the only pair found in *both* tables of values. It represents the only day when the girls will have the same amount of money.

Method 2: Use a Graph

The girls draw graphs on the same grid. This enables them to compare the linear relationships for their savings.



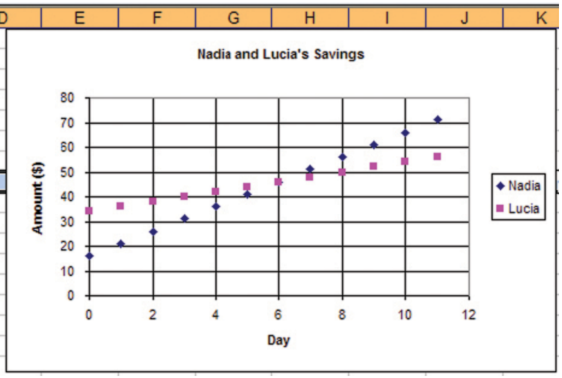
Why are the points not joined on this graph?

The intersection point of the two relationships is (6, 46). This means the girls will have the same amount of money, $46, on day 6.

Method 3: Use a Spreadsheet

In a spreadsheet, the girls enter the headings and amounts shown. Then, they use the spreadsheet's graphing features.

	A	B	C
1	Day	Nadia	Lucia
2	0	$16	$34
3	1	$21	$36
4	2	$26	$38
5	3	$31	$40
6	4	$36	$42
7	5	$41	$44
8	6	$46	$46
9	7	$51	$48
10	8	$56	$50
11	9	$61	$52
12	10	$66	$54
13	11	$71	$56
14			
15			
16			
17			



The table of values shows that the girls will have the same amount of money on day 6. The point of intersection on the graph is (6, 46). They will both have $46 on day 6.

Your Turn

Davidee earns $40 plus $10 per hour. Carmen earns $50 plus $8 per hour.

a) Represent the linear system relating the earnings numerically and graphically.

b) Identify the solution to the linear system and explain what it represents.

Example 2 Solve a Linear System Graphically

a) Consider the system of linear equations $2x + y = 2$ and $x - y = 7$. Identify the point of intersection of the lines by graphing.

b) Verify the solution.

Solution

How will the form of each equation help you choose your method of graphing?

a) Graph the equations together.

Method 1: Use Slope-Intercept Form

Rearrange each equation into slope-intercept form by isolating y. Identify the y-intercept and slope to draw the graph.

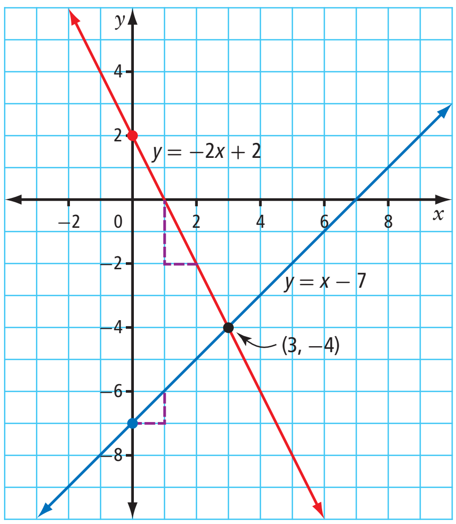
$$\begin{aligned} 2x + y &= 2 \\ 2x + y - 2x &= 2 - 2x \\ y &= 2 - 2x \\ y &= -2x + 2 \end{aligned}$$

$$\begin{aligned} x - y &= 7 \\ x - y + y &= 7 + y \\ x &= 7 + y \\ x - 7 &= 7 + y - 7 \\ x - 7 &= y \\ y &= x - 7 \end{aligned}$$

The y-intercept is 2.
The slope is -2.

The y-intercept is -7.
The slope is 1.

The solution is shown on the graph. It is the point of intersection, $(3, -4)$.



Method 2: Use *x*-Intercepts and *y*-Intercepts

Determine the x-intercept and y-intercept of the line $2x + y = 2$.

x-intercept: $y = 0$	y-intercept: $x = 0$
$2x + y = 2$	$2x + y = 2$
$2x + 0 = 2$	$2(0) + y = 2$
$2x = 2$	$y = 2$
$x = 1$	

If given the equation of a line, how can you determine its intercepts?

For the equation $2x + y = 2$, the x-intercept is 1 and the y-intercept is 2.

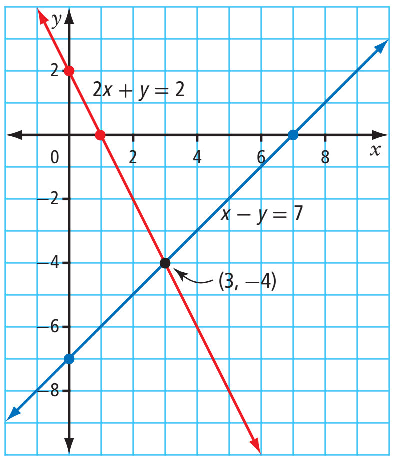
Determine the intercepts of the line $x - y = 7$.

x-intercept: $y = 0$	y-intercept: $x = 0$
$x - y = 7$	$x - y = 7$
$x - 0 = 7$	$0 - y = 7$
$x = 7$	$y = -7$

For the equation $x - y = 7$, the x-intercept is 7 and the y-intercept is -7.

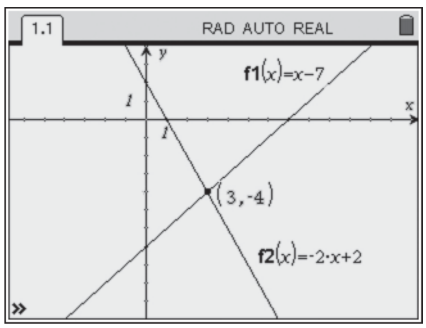
For each equation, plot the x-intercept and y-intercept; then, join the points.

The lines intersect at the point $(3, -4)$. So, the solution to the linear system is $(3, -4)$.



Method 3: Use Technology

Graph each equation using technology. Adjust the dimensions of the graph until you see both intercepts of each line, as well as the point of intersection. Then, use the intersection feature to find the solution.



The intersection point $(3, -4)$ is the solution to the linear system.

b) To verify that (3, −4) is the solution to the linear system $2x + y = 2$ and $x - y = 7$, use a different representation than the method of solving.

Method 1: Substitute Using Paper and Pencil

Verify the solution (3, −4) by substituting the values of x and y into each equation.

In $2x + y = 2$:

Left Side	Right Side
$2x + y$	2
$= 2(3) + (-4)$	
$= 6 - 4$	
$= 2$	

Left Side = Right Side

In $x - y = 7$:

Left Side	Right Side
$x - y$	7
$= 3 - (-4)$	
$= 3 + 4$	
$= 7$	

Left Side = Right Side

Since the ordered pair (3, −4) satisfies both equations, it is the solution to the linear system.

Method 2: Create a Table of Values Using Technology

Enter the equations of the lines and generate a table of values.

x	f1(x):= -2*x+2	f2(x):= x−7
0	2	-7
1	0	-6
2	-2	-5
3	-4	-4
4	-6	-3
5	-8	-2

When $x = 3$, both equations have the same value for y of −4. So, the point (3, −4) is the solution to the linear system.

Your Turn

Verify by graphing and one other way that (3, −2) is the solution to the system of linear equations $x - 3y = 9$ and $2x + y = 4$.

Example 3 Connect a Solution and a Graph

Guy solved the linear system $x - 2y = 12$ and $3x - 2y = 4$. His solution is (2, −5). Verify whether Guy's solution is correct. Explain how Guy's results can be illustrated on a graph.

Solution

Substitute $x = 2$ and $y = -5$ into each equation. Evaluate each side to determine whether the values satisfy both equations.

In $x - 2y = 12$:

Left Side	Right Side
$x - 2y$	12
$= 2 - 2(-5)$	
$= 2 + 10$	
$= 12$	

Left Side = Right Side

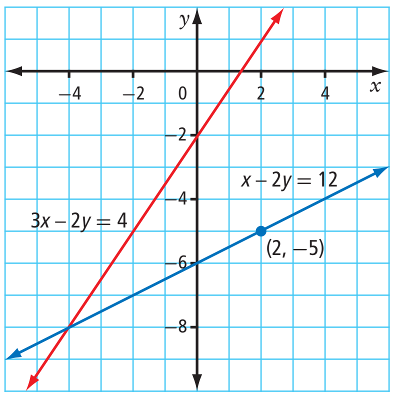
In $3x - 2y = 4$:

Left Side	Right Side
$3x - 2y$	4
$= 3(2) - 2(-5)$	
$= 6 + 10$	
$= 16$	

Left Side ≠ Right Side

The given values satisfy the first equation but not the second equation. Since both equations do not result in true statements, the point (2, −5) is not the solution to this linear system.

The point (2, −5) is not a solution to the given linear system. So, a graph of this system will not have a point of intersection at (2, −5). The point (2, −5) is on one of the lines, not *both* lines.



Your Turn

For each system of linear equations, verify whether the given point is a solution. Explain what the results would show on a graph.

a) $3x - y = 2$
$x + 4y = 32$
(2, 5)

b) $2x + 3y = -12$
$4x - 3y = -6$
(−3, −2)

Example 4 Solve a Problem Involving a Linear System

The Skyride is a red aerial tram that carries passengers up Grouse Mountain in Vancouver, BC. The Skyride travels from an altitude of about 300 m to an altitude of 1100 m. The tram can make the trip up or down in 5 min and can carry 100 passengers.

There is also a blue tram that can carry 45 passengers. This tram takes approximately 8 min to travel up or down the mountain. Each tram travels at a constant speed.

a) Create a graph to represent the altitudes of the trams if the red tram starts at the top and the blue tram starts at the base.

b) Explain the meaning of the point of intersection.

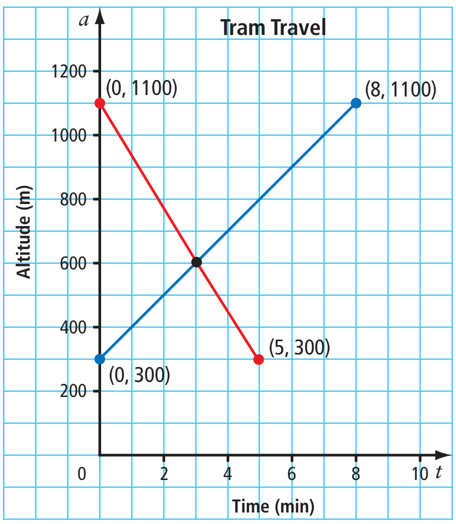
Solution

a) Organize the information before graphing.

Tram	Start		End		Representation on a Graph
	Time	Altitude	Time	Altitude	
Red	0 min	1100 m	5 min	300 m	Line segment joining the points (0, 1100) and (5, 300)
Blue	0 min	300 m	8 min	1100 m	Line segment joining the points (0, 300) and (8, 1100)

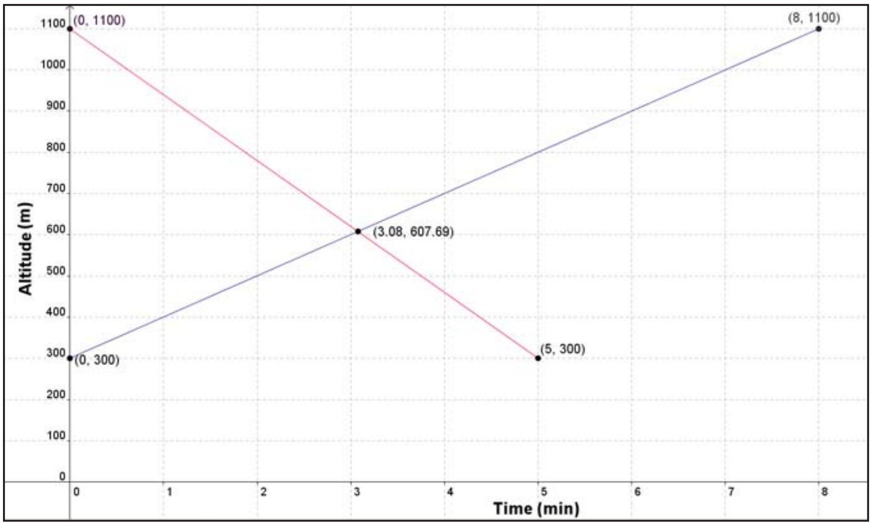
Method 1: Use Paper and Pencil

Label time from 0 min to 10 min on the horizontal axis. Label altitude on the vertical axis, up to 1200 m. Graph a line segment for each tram using the start and end points.



Method 2: Use Technology

Plot the start and end points for the travel of each tram. Use the "segment between two points" or equivalent feature to connect the points to show the continuous travel for each tram.



Use the intersection feature to determine the solution to the linear system.

b) At the point of intersection, the two trams will have the same altitude at the same time. The lines appear to intersect at approximately (3, 600). Therefore, after about 3 min, the two trams will pass each other at about 600 m in altitude.

The red tram travels faster, so it will travel farther than the blue tram in the same time interval. The red tram starts at the top, so the expected solution will be closer to the base of the mountain.

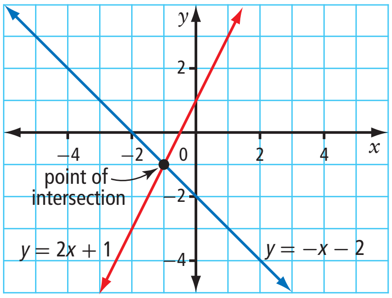
Your Turn

Eric works on the 23rd floor of a building. It takes Eric 90 s to walk down the stairs to the 14th floor. Nathan works on the 14th floor and needs to go up to the 30th floor. He knows it will take 40 s by elevator if the elevator makes no other stops.

Suppose both men leave their offices at the same time. Create a graph to model their travel. What does the point of intersection represent?

Key Ideas

- Systems of linear equations can be modelled numerically, graphically, or algebraically.
- The solution to a linear system is a pair of values that occurs in each table of values, an intersection point of the lines on a graph, or an ordered pair that satisfies each equation.
- One way to solve a system of linear equations is to graph the lines and identify the point of intersection on the graph.



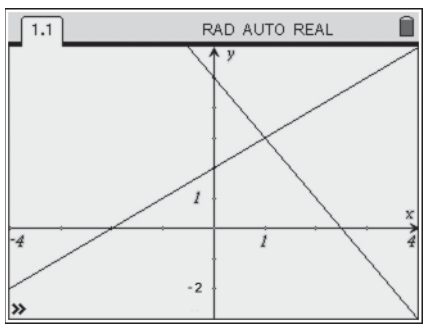
- A solution to a system of linear equations can be verified using several methods:
  - Substitute the value for each variable and evaluate the equations.
  - Create a graph and identify the point of intersection.
  - Create tables of values and identify the pair of values that occurs in each table.

Check Your Understanding

Practise

1. One linear system is shown in the table of values, and another in the graph. Do the two systems have the same solution? Justify your answer.

x	f1(x):= 4−x	f2(x):= 2*x+1
0.	4.	1.
1.	3.	3.
2.	2.	5.
3.	1.	7.
4.	0.	9.



2. John uses technology to check whether (5.2, 3) is the solution to the linear system $7x - 2y = 30.4$ and $4x + y = 25.1$. The results are shown on the screen. Is John's solution correct? Explain.

Define x=5.2	*Done*
Define y=3	*Done*
$7\cdot x-2\cdot y$	30.4
$4\cdot x+y$	23.8

3. a) Represent the system of linear equations $y = 2x + 6$ and $y = 3x$ using a table of values and a graph.

b) Explain how you can identify the solution to the linear system from your table of values and your graph.

c) Verify by substitution that the solution you found is correct.

4. Consider the system of linear equations $y = -x + 4$ and $y = \frac{1}{2}x - 5$.

a) Show how a table of values can be used to solve the linear system.

b) Show how the linear system can be solved using a graphical representation.

c) Explain how the solution is related to the original equations.

5. Is each given point a solution to the system of linear equations? Explain.

a) $y = 3x - 5$
$y = 11 - x$
(4, 7)

b) $4x + 3y = 5$
$x + 4y = 13$
(−1, 3)

c) $2x - 3y = 18$
$x + 2y = -26$
(−6, −10)

d) $12x - 3y = 7$
$y = 4.5x - 3$
(1.2, 2.4)

6. On grid paper, graph each system of linear equations. What is the solution for each linear system?

a) $y = -2x + 5$
$y = x - 4$

b) $4x - y = -8$
$2x + 3y = -18$

7. Solve each system of linear equations graphically.

a) $y = 2x - 10$
$y = -3x + 8$

b) $y = \frac{1}{2}x - 5$
$y = -\frac{4}{3}x + 1$

c) $2x + y = 24$
$2x + 5y = 50$

d) $x - 2y = -18$
$3x + 4y = -12$

8. Solve each linear system graphically. Then, verify your solution.

a) $y = 0.5x + 4$
$y = 0.8x + 1$

b) $2x - 5y = 40$
$-6x + 5y = -60$

9. Is each given point a solution to the system of linear equations? Explain what the results would show on a graph of the linear system.

a) $3x - y = 2$
$x + 4y = 22$
$(2, 5)$

b) $2x + 3y = -12$
$4x - 3y = -6$
$(-3, -2)$

10. Brad and Sharon are collecting money from family and friends for a local charity. Brad has \$35 and plans to add \$5 each day. Sharon does not have any money yet. She plans to collect \$12 each day.

a) Represent the donations Brad and Sharon are collecting using a table of values and a graph.

b) What is the solution of the linear system? What does it represent?

Apply

11. Maya makes bead necklaces for a craft fair. It costs her \$2.50 to make each necklace. She needs to pay \$49 for a table at the craft fair. She plans to sell each necklace for \$6. Maya models her costs and revenue with the following equations:
Costs: $y = 2.5x + 49$
Revenue: $y = 6x$
In the equations, x represents the number of necklaces and y represents the amount, in dollars.

a) Create a graph of the linear system.

b) How many necklaces must Maya sell in order to break even?

c) Explain how to use the graph to determine the profit Maya will make if she sells 20 necklaces.

12. Consider the system of linear equations $y = \frac{1}{2}x - 5$ and $y = -\frac{1}{3}x + 11$.

a) Sketch a graph of the linear system on grid paper. Estimate the solution.

b) Use technology to solve the linear system by graphing.

c) Discuss the advantages and disadvantages of each method of solving systems of linear equations.

13. Can you solve the linear system involving $f_1(x) = 41 - 2x$ and $f_2(x) = 14 + 4x$ using the table of values shown? Explain.

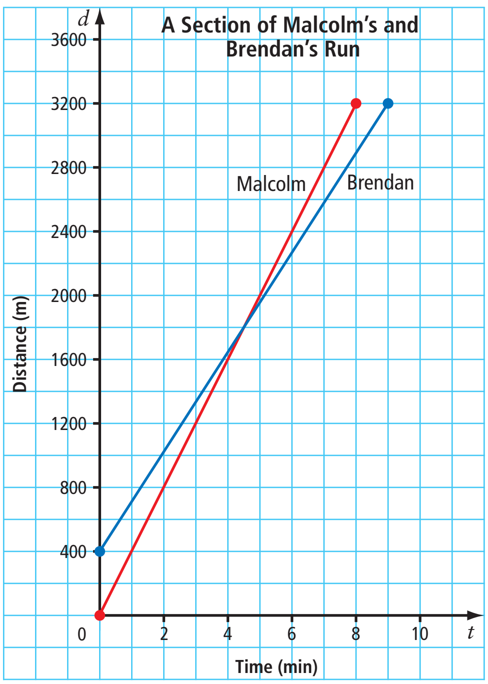
x	f1(x):= 41−2*x	f2(x):= 14+4*x
-1.	43.	10.
0.	41.	14.
1.	39.	18.
2.	37.	22.
3.	35.	26.
4.	33.	30.
5.	31.	34.
6.	29.	38.

14. Kayla and Sam sketch a graph to solve the system of linear equations $3x - 2y = -8$ and $4x - 3y = -9$. Sam completes his graph first and tells Kayla, "You need to graph extra carefully to solve this system." Solve the linear system by graphing and verify your solution. Why might Sam be justified in making his comment to Kayla?

15. The Calgary Marathon has been an annual event since 1971. It attracts participants from Canada and the United States. Malcolm and Brendan are training for the marathon. The graph shows a section of one of their long-distance runs.

a) Describe the part of their run represented in the graph.

b) Why can a system of linear equations represent this part of their run?



16. **Unit Project** Two groups of ducks are leaving a field and heading for a water source 50 km away. The green-winged teals leave 25 min before the canvasback ducks. Green-winged teals fly at a speed of 48 km/h.

a) How far do the green-winged teals fly during the 25 min?

b) Canvasback ducks fly at a speed of 115 km/h. The distance, d, in kilometres, travelled by each species is related to time, t, in hours, by the following equations:

Green-winged teals: $d = 48t + 20$
Canvasback ducks: $d = 115t$

What does time, $t = 0$ represent? Justify your answer. Then, sketch a graph of the system of linear equations.

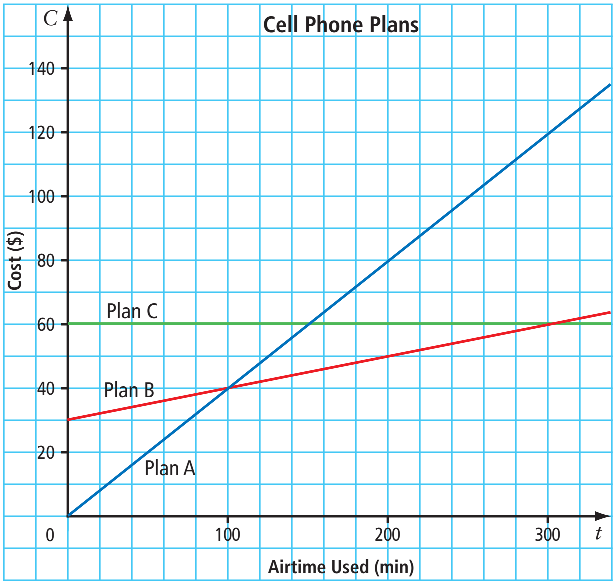
c) Use the graph to describe the trip to the water source for the two groups of ducks. Explain your reasoning.

17. One hot-air balloon is 100 m above ground. It rises at a constant rate and reaches a height of 1000 m in 16 min. Another hot-air balloon is 1600 m above ground. It descends at a constant rate to ground level in 20 min. Create a graph to represent the travel of the balloons. What does the point of intersection represent?

18. Sarah and George are both using the road from Qamani'tauq to the bridge over the Prince River in Nunavut. The road is 35 km long. Sarah drives her ATV at a constant speed from Qamani'tauq to the bridge. She takes 30 min. George drives his snowmobile at a constant speed from the bridge to Qamani'tauq. He takes 23 min. Create a graph to represent each person's distance from the bridge. What does the point of intersection represent?

Extend

19. The graph shows the costs for three different cell phone plans. Describe a situation in which a user would benefit most from each plan as compared with the other plans.



20. A truck travels along a highway. The truck's speed can be modelled with the equation $s = 1.5t + 5$. In the equation, s represents speed, in metres per second, and t represents time, in seconds. As the truck reaches a parked car, the car begins to move ahead. The car's speed can be modelled with the equation $s = 2.3t$. How many seconds does the car travel until its speed is the same as the truck's speed?

Create Connections

21. Describe a situation in your life that could be represented with a system of linear equations. Sketch a graph of the linear system. Explain what the point of intersection would represent.

22. How does *solving* a system of linear equations differ from *verifying* a solution to a system of linear equations? Provide an example.

23. Consider the system of linear equations $Ax + By + C = 0$ and $Dx + Ey + F = 0$. For each set of criteria, describe the lines. Justify your answers.

a) $A = D, B = E, C \neq F$

b) $A = D^{-1}, B = -E^{-1}, C = F$

c) $A \neq D, B = E, C = F$

d) $A = D, B = E, C = F$