5.4 Factoring Special Trinomials

Did You Know?

Quilting has often been a way to unite people from different countries and cultures. The quilt shown here was part of a collection of quilts made by the Canadian Red Cross during WWII. These quilts were sent to families in Britain who had been displaced because of the war.
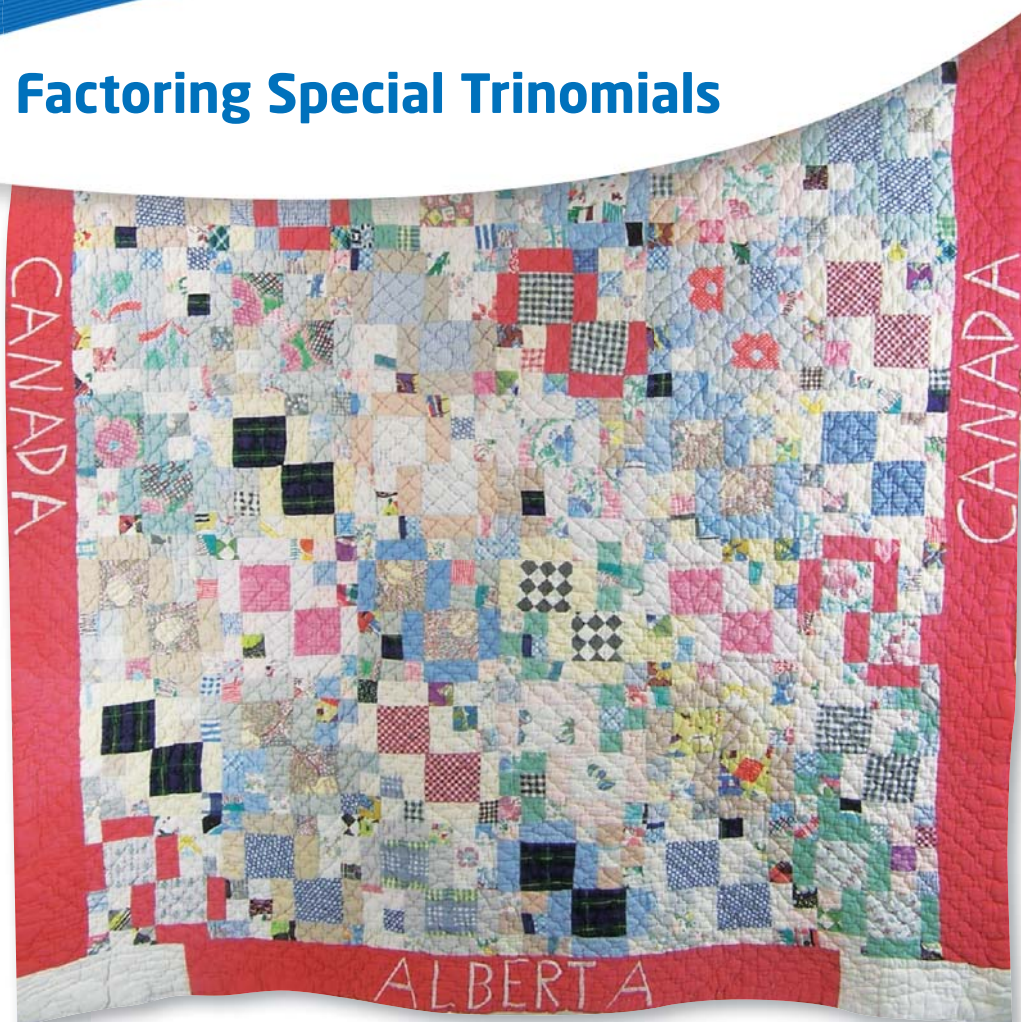
difference of squares
- an expression of the form $a^2 - b^2$ that involves the subtraction of two squares
- for example, $x^2 - 4$, $y^2 - 25$

**Materials**
- centimetre grid paper
- scissors

Patchwork quilts are made of square pieces of fabric sewed together to form interesting patterns. How could you relate these squares to polynomials and their factors?

Some polynomials, like perfect square trinomials and **differences of squares**, follow patterns that allow you to recognize a type of factoring method to use.

## Investigate Factoring Differences of Squares

1. Cut a 10-cm by 10-cm square out of a piece of centimetre grid paper.
    a) What is the area of the square?
    b) How did you calculate this area?

2. Cut a 4-cm by 4-cm piece from the corner of the square.

   a) What is the area of this cutout piece?

   b) How did you calculate this area?
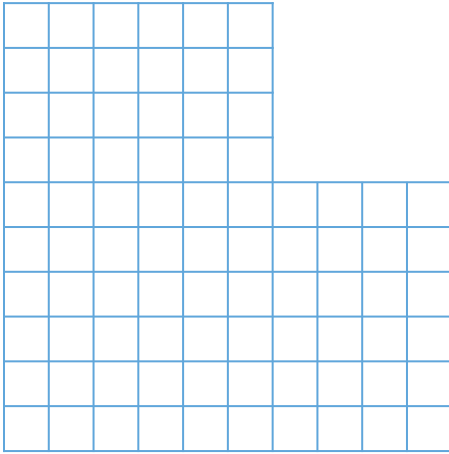
3. a) Calculate the area of the remaining paper.

   b) How did you calculate this area?

   c) Are there other methods you could use to calculate this area? Explain.

4. Make one cut to the irregular shape that remains, so that you can rearrange it to form a rectangle.

   a) What are the dimensions of the rectangle?

   b) What is the area of the rectangle?

   c) How is the area of the shape in step 3 related to the area of this rectangle?

5. Repeat step 1 to step 4 for two additional squares of different sizes.

   a) Can each irregular shape always be rearranged into a rectangle? Compare your answer with a partner's.

   b) List the dimensions of each rectangle.

   c) Explain how the area of the cutout shape relates to the area of the rectangle.

6. a) Write an algebraic expression to represent the area remaining when a square of area 25 cm² is removed from a square of area x^2 square centimetres.

   b) If the resulting shape is rearranged into a rectangle, what are its dimensions?

   c) Explain the relationship between your answers to parts a) and b).

   **d)** Write an equation showing this relationship.

7. Reflect and Respond The diagram shows what remains when a square of dimensions b by b is removed from a square of dimensions a by a.
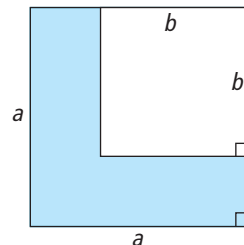
a) Write an expression to represent the area of the remaining shaded shape.

b) The shape is rearranged to form a rectangle. What are the dimensions of the rectangle?

c) Write an expression to represent the area of the rectangle.

d) Write an equation to show the relationship between the area of the remaining shape and the area of the rectangle.

8. a) What are the patterns you observed from cutting out and rearranging the squares?

b) What conclusions can you make about subtracting the area of a smaller square from the area of a larger square?

Link the Ideas

When you cut a square out of a square, the area of the remaining shape is a difference of two squares. When you cut and rearrange this paper into a rectangle, you can write the area as a product of its dimensions.
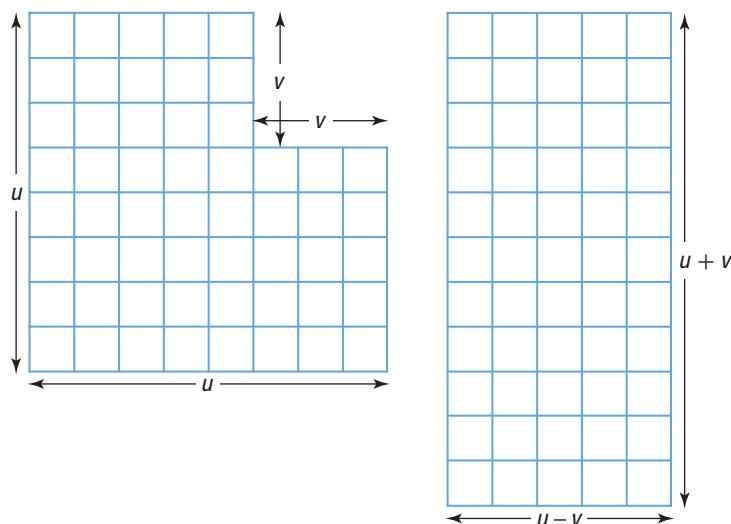
You will find patterns helpful in factoring polynomials with special products. These include differences of squares and perfect square trinomials.

Difference of Squares

When you multiply the sum and the difference of two terms, the product will be a difference of squares.

$$(u + v)(u - v) = (u)(u - v) + (v)(u - v)$$
$$= (u)(u) - (u)(v) + (v)(u) + (v)(-v)$$
$$= u^2 - uv + uv - v^2$$
$$= u^2 - v^2$$

In a difference of squares
- the expression is a binomial
- the first term is a perfect square: u^2
- the last term is a perfect square: v^2
- the operation between the two terms is subtraction

A difference of squares, $u^2 - v^2$, can be factored into $(u + v)(u - v)$.

Perfect Square Trinomial

When you square a binomial, the result is a perfect square trinomial.

$$(x + 5)^2 = (x + 5)(x + 5)$$
$$= x(x + 5) + 5(x + 5)$$
$$= x^2 + 5x + 5x + 25$$
$$= x^2 + 10x + 25$$

In a perfect square trinomial
- the first term is a perfect square: x^2
- the last term is a perfect square: 5^2
- the middle term is twice the product of the square root of the first term and the square root of the last term: $(2)(x)(5) = 10x$
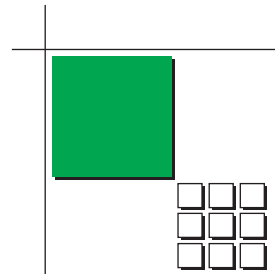
Example 1 Factor a Difference of Squares

Factor each binomial, if possible.
a) $x^2 - 9$ **b)** $16c^2 + 25a^2$
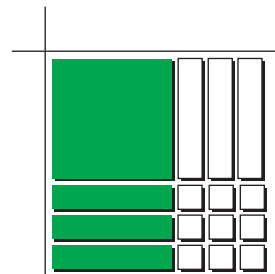**c)** $m^2 + 16$ **d)** $7g^3h^2 - 28g^5$

Solution

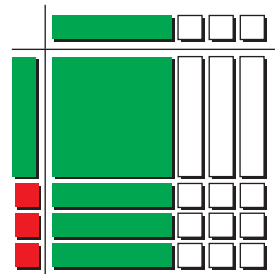**a) Method 1: Use Algebra Tiles**

Create an algebra tile model to represent $x^2 - 9$.

Add three positive x-tiles and three negative x-tiles to represent the middle term.

The dimensions represent the factors of $x^2 - 9$.

The factors are $x - 3$ and $x + 3$.
Therefore, $x^2 - 9 = (x - 3)(x + 3)$.
Check:
Multiply.
$$(x - 3)(x + 3) = x(x + 3) - 3(x + 3)$$
$$= x^2 + 3x - 3x - 9$$
$$= x^2 - 9$$

Method 2: Factor by Grouping
$$x^2 - 9 = x^2 - 3x + 3x - 9$$
$$= (x^2 - 3x) + (3x - 9)$$
$$= x(x - 3) + 3(x - 3)$$
$$= (x - 3)(x + 3)$$

The middle term must be included. Add in the zero pairs.

Method 3: Factor as a Difference of Squares

The binomial $x^2 - 9$ is a difference of squares.

The first term is a perfect square: x^2

The last term is a perfect square: 3^2

The operation is subtraction.

$x^2 - 9 = x^2 - 3^2$
$ = (x - 3)(x + 3)$

b) You can write $-16c^2 + 25a^2$ as $25a^2 - 16c^2$.

The binomial $25a^2 - 16c^2$ is a difference of squares.

The first term is a perfect square: $(5a)^2$

The last term is a perfect square: $(4c)^2$

The operation is subtraction.

$25a^2 - 16c^2 = (5a)^2 - (4c)^2$
$ = (5a - 4c)(5a + 4c)$

Check:

Multiply.

$(5a - 4c)(5a + 4c) = (5a)(5a + 4c) + (-4c)(5a + 4c)$
$ = 25a^2 + 20ac - 20ac - 16c^2$
$ = 25a^2 - 16c^2$
$ = -16c^2 + 25a^2$

c) The binomial $m^2 + 16$ can be written as a trinomial where the middle term is $0m$.

$m^2 + 0m + 16$

To factor this expression, you need to find two integers with
- a product of 16   • a sum of 0

Since the product is positive, both integers need to be either positive or negative.

If both integers are either positive or negative, a sum of 0 is not possible. Therefore, the binomial $m^2 + 16$ cannot be factored over the integers.

d) First, factor out the GCF from $7g^3h^2 - 28g^5$.

$7g^3h^2 - 28g^5 = 7g^3(h^2 - 4g^2)$

The binomial is a difference of squares.

The first term is a perfect square: h^2

The last term is a perfect square: $(2g)^2$

The operation is subtraction.

$7g^3h^2 - 28g^5 = 7g^3(h^2 - 4g^2)$
$ = 7g^3[h^2 - (2g)^2]$
$ = 7g^3(h - 2g)(h + 2g)$

Check:

Multiply.

$7g^3(h - 2g)(h + 2g) = 7g^3[h(h + 2g) - 2g(h + 2g)]$
$ = 7g^3(h^2 + 2gh - 2gh - 4g^2)$
$ = 7g^3(h^2 - 4g^2)$
$ = 7g^3h^2 - 28g^5$

Factor each binomial, if possible.
a) $49a^2 - 25$
b) $125x^2 - 40y^2$
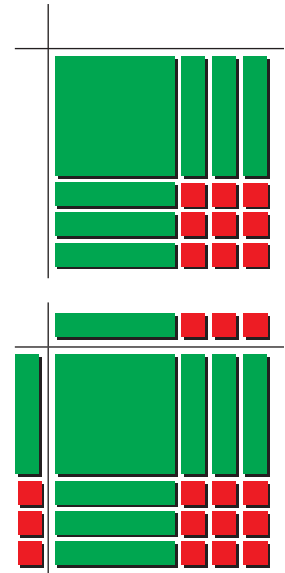**c)** $9p^2q^2 - 25$

Example 2 Factor Perfect Square Trinomials

Factor each trinomial, if possible.
a) $x^2 + 6x + 9$ **b)** $2x^2 - 44x + 242$ **c)** $c^2 - 12x - 36$

Solution

a) Method 1: Use Algebra Tiles
Create an algebra tile model to
represent $x^2 + 6x + 9$.

The dimensions represent the factors
of $x^2 + 6x + 9$.

The factors are $x + 3$ and $x + 3$.
Therefore, $x^2 + 6x + 9 = (x + 3)(x + 3)$
$\qquad\qquad\qquad\quad = (x + 3)^2$

Check:
Multiply.
$(x + 3)(x + 3) = x(x + 3) + 3(x + 3)$
$\qquad\qquad\quad = x^2 + 3x + 3x + 9$
$\qquad\qquad\quad = x^2 + 6x + 9$

Method 2: Factor by Grouping
$x^2 + 6x + 9 = (x^2 + 3x) + (3x + 9)$
$\qquad\qquad\quad = x(x + 3) + 3(x + 3)$
$\qquad\qquad\quad = (x + 3)(x + 3)$
$\qquad\qquad\quad = (x + 3)^2$

Method 3: Factor as a Perfect Square Trinomial

The trinomial $x^2 + 6x + 9$ is a perfect square.

The first term is a perfect square: x^2

The last term is a perfect square: 3^2

The middle term is twice the product of the square root of the first term and the square root of the last term: $(2)(x)(3) = 6x$

The trinomial is of the form $(ax)^2 + 2abx + b^2$.

$$x^2 + 6x + 9 = (x + 3)(x + 3)$$
$$= (x + 3)^2$$

b) First, factor out the GCF from $2x^2 - 44x + 242$.

$$2x^2 - 44x + 242 = 2(x^2 - 22x + 121)$$

The first term in the brackets is a perfect square: x^2

The last term in the brackets is a perfect square: 11^2

The middle term is twice the product of the square root of the first term and the square root of the last term: $(2)(x)(11) = 22x$

The trinomial is of the form $(ax)^2 - 2abx + b^2$.

$$2x^2 - 44x + 242 = 2(x^2 - 22x + 121)$$
$$= 2(x - 11)(x - 11)$$
$$= 2(x - 11)^2$$

Check:

Multiply.

$$2(x - 11)(x - 11) = 2[x(x - 11) - 11(x - 11)]$$
$$= 2(x^2 - 11x - 11x + 121)$$
$$= 2(x^2 - 22x + 121)$$
$$= 2x^2 - 44x + 242$$

c) The trinomial $c^2 - 12x - 36$ is not a perfect square.

The first and last terms are perfect squares.

The middle term is twice the product of the square root of the first term and the square root of the last term.

However, the trinomial is not of the form $(ax)^2 + 2abx + b^2$ or $(ax)^2 - 2abx + b^2$.

Therefore, the trinomial cannot be factored over the integers.

Your Turn

Factor each trinomial, if possible.

a) $x^2 - 24x + 144$

b) $y^2 - 18y - 81$

c) $3b^2 + 24b + 48$

Key Ideas

- Some polynomials are the result of special products. When factoring, you can use the pattern that formed these products.

Difference of Squares:

The expression is a binomial.

The first term is a perfect square.

The last term is a perfect square.

The operation between the terms is subtraction.

$$x^2 - 25 = x^2 - 5^2$$
$$= (x - 5)(x + 5)$$

Perfect Square Trinomial:

The first term is a perfect square.

The last term is a perfect square.

The middle term is twice the product of the square root of the first term and the square root of the last term.

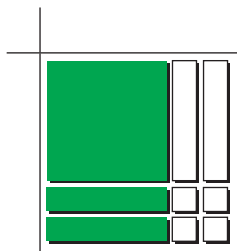The trinomial is of the form $(ax)^2 + 2abx + b^2$ or $(ax)^2 - 2abx + b^2$.

$$x^2 + 16x + 64 = x^2 + 8x + 8x + 64$$
$$= x(x + 8) + 8(x + 8)$$
$$= (x + 8)(x + 8)$$

Check Your Understanding

Practise

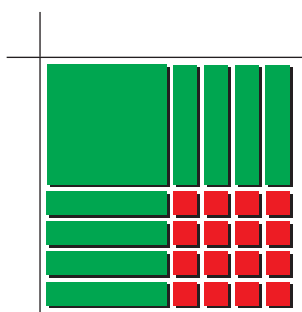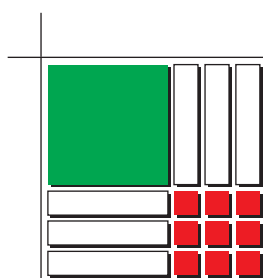**1.** Identify the factors of the polynomial shown by each algebra tile model.

a)

b)

c)

d)

2. Determine each product.

a) $(x - 8)(x + 8)$

b) $(2x + 5)(2x - 5)$

c) $(3a - 2b)(3a + 2b)$

d) $3(t - 5)(t + 5)$

3. What is each product?

a) $(x + 3)^2$

b) $(3b - 5a)^2$

c) $(2h + 3)^2$

d) $5(x - 2y)^2$

4. Identify the missing values for a difference of squares or a perfect square trinomial.

a) $\blacksquare - y^2 = (\blacksquare - y)(m + \blacksquare)$

b) $16r^6 - \blacksquare = (\blacksquare - \blacksquare)(\blacksquare + 9)$

c) $x^2 - 12x + \blacksquare = (\blacksquare - 6)^2$

d) $4x^2 + \blacksquare + \blacksquare = (\blacksquare + 5)^2$

e) $\blacksquare + \blacksquare + 49 = (5x + \blacksquare)(\blacksquare + \blacksquare)$

5. Factor each binomial, if possible.

a) $x^2 - 16$

b) $b^2 - 121$

c) $w^2 + 169$

d) $9a^2 - 16b^2$

e) $36c^2 - 49d^2$

f) $h^2 + 36f^2$

g) $121a^2 - 124b^2$

h) $100 - 9t^2$

6. Factor each trinomial, if possible.

a) $x^2 + 12x + 36$

b) $x^2 + 10x + 25$

c) $a^2 - 24a - 144$

d) $m^2 - 26m + 169$

e) $16k^2 - 8k + 1$

f) $49 - 14m + m^2$

g) $81u^2 + 34u + 4$

h) $36a^2 + 84a + 49$

7. Factor completely.

a) $5t^2 - 100$

b) $10x^3y - 90xy$

c) $4x^2 - 48x + 36$

d) $18x^3 + 24x^2 + 8x$

e) $x^4 - 16$

f) $x^4 - 18x^2 + 81$

Apply

8. Determine two values of n that allow each polynomial to be a perfect square trinomial. Then, factor.

a) $x^2 + nx + 25$

b) $a^2 + na + 100$

c) $25b^2 + nb + 49$

d) $36t^2 + nt + 121$

9. Each of the following polynomials cannot be factored over the integers. Why not?

a) $25a^2 - 16b$

b) $x^2 - 7x - 12$

c) $4r^2 - 12r - 9$

d) $49t^2 + 100$

10. **(Unit Project)** Use models or diagrams to show what happens to the middle terms when you multiply two factors that result in a difference of squares. Include at least two specific examples.

11. Many number tricks can be explained using factoring. Use $a^2 - b^2 = (a - b)(a + b)$ to make the following calculations possible using mental math.

a) $19^2 - 9^2$

b) $28^2 - 18^2$

c) $35^2 - 25^2$

d) $5^2 - 25^2$

12. **(Unit Project)**

a) Use models or diagrams to show the squaring of a binomial. Include at least two specific examples.

b) Create a rule for squaring any binomial. Show how your rule relates to your models or diagrams.

13. Zoë wants to construct a patio in the corner of her property. The area of her square property has a side length represented by x metres. The patio will take up a square area with a side length represented by y metres. Write an expression, in factored form, to represent the remaining area of the property.

14. The diagram shows two concentric circles with radii r and $r + 4$.

a) Write an expression for the area of the shaded region.

b) Factor this expression completely.

c) If $r = 6$ cm, calculate the area of the shaded region. Give your answer to the nearest tenth of a square centimetre.
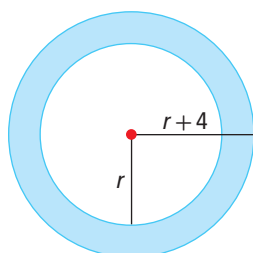
15. An object is reduced or enlarged uniformly in all dimensions. The print shown is a watercolour painting called *August Chinook* by Gena LaCoste of Medicine Hat, Alberta. This print is going to be enlarged by a factor of 3. The side length of the original can be represented by $(2x - 3)$ cm.

   a) Use your understanding of differences of squares to write an expression that represents the difference in the areas of the original print and the enlargement.

   b) Multiply this expression to write it in the form $ax^2 + bx + c$.

   c) Verify that your expressions in parts a) and b) are correct by substituting a value for x.

16. Explain how the diagram shows a difference of squares.
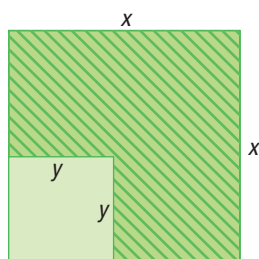
17. The area of a square can be given by $49 - 28x + 4x^2$, where x represents a positive integer. Write a possible expression for the perimeter of the square.

Area = $49 - 28x + 4x^2$
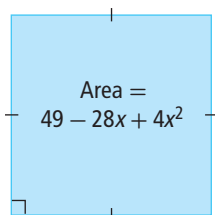
18. The circular area of the painted butterfly drum can be represented by the expression $(9x^2 + 30x + 25)\pi$. Determine an expression for the smallest diameter the drum could have.

Traditional Tlingit hand drums are used in ceremony, cultural and social events, and as artwork. Traditional drums should always be handled with respect following appropriate protocol.

19. State whether the following equations are *sometimes, always*, or *never* true. Explain your reasoning.

a) $a^2 - 2ab - b^2 = (a - b)^2, b \neq 0$

b) $a^2 + b^2 = (a + b)(a + b)$

c) $a^2 - b^2 = a^2 - 2ab + b^2$

d) $(a + b)^2 = a^2 + 2ab + b^2$

20. Rahim and Kate are factoring $16x^2 + 4y^2$. Who is correct? Explain your reasoning.
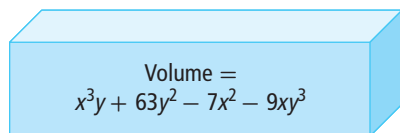

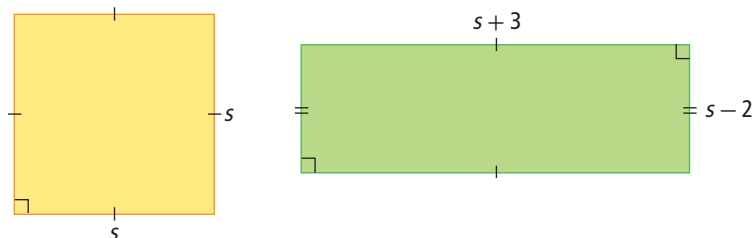Rahim
$16x^2 + 4y^2 = 4(4x^2 + y^2)$

Kate
$16x^2 + 4y^2 = 4(4x^2 + y^2)$
$= 4(2x + y)(2x - y)$

Extend

21. The volume of a rectangular prism is $x^3y + 63y^2 - 7x^2 - 9xy^3$. Determine expressions for the dimensions of the prism.

Volume =
$x^3y + 63y^2 - 7x^2 - 9xy^3$

22. The area of the square shown is $16x^2 - 56x + 49$. What is the area of the rectangle in terms of x?

s

s

$s + 3$

$s - 2$

23. a) The difference of squares of two numbers is the same as their sum. What integers satisfy this condition? Show how you determined your answer.

b) Based on your observations in part a), identify two integers from 11 to 20 which have a difference of squares that can be expressed as the sum of the integers.

Create Connections

24. a) If $x^2 + bx + c$ is a perfect square, how are b and c related?

b) If $ax^2 + bx + c$ is a perfect square, how are a, b, and c related?

25. Use two ways to show that $a^2 - b^2 = (a - b)(a + b)$.

26. What is the difference in factoring $x^2 + 2bx + b^2$ and $x^2 - 2bx + b^2$?

27. To determine the product of two numbers that differ by 2, square their average and then subtract 1. Use this method to find the following products.

$(29)(31) = \blacksquare$
$(59)(61) = \blacksquare$

a) Explain this method using a difference of squares.

b) Develop a similar method for multiplying two numbers that differ by 6.

c) Explain your method from part b) using a difference of squares.