CHAPTER 1
Limits and Their Properties

CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Exercises

1. Precalculus: (20 ft/sec)(15 seconds) = 300 feet

2. Calculus required: velocity is not constant
   Distance $\approx$ (20 ft/sec)(15 seconds) = 300 feet

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

4. Precalculus: rate of change = slope = 0.08

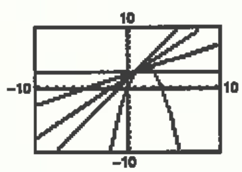
5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$

6. Calculus required: Area $\approx \frac{1}{2}\pi(3)^2 = \frac{9}{2}\pi$

7. Precalculus: Volume = (2)(4)(3) = 24

8. Precalculus: Volume $= \pi(3)^2 6 = 54\pi$

9. (a)



   (b) The graphs of y_2 are approximations to the tangent line to y_1 at $x = 1$.

   (c) The slope is approximately 2. For a better approximation make the list numbers smaller:
   {0.2, 0.1, 0.01, 0.001}

10. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

    Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

    (b) You could improve the approximation by using more rectangles.

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

    (b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

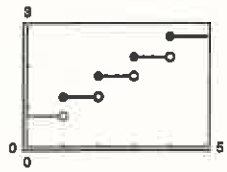
    $\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

    (c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1. $C(t) = 0.75 - 0.50[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$\lim_{t\to 3.5} C(t) = 2.25$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t\to 3} C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t = 3$.

2. $\lim_{x\to 8} f(x) = 25$ means that the values of $f(x)$ approach 25 as x approaches 8, from both sides of 8.

3.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$\lim_{x\to 2} \frac{x-2}{x^2-x-2} \approx 0.3333$ (Actual limit is $\frac{1}{3}$.)

4.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 2} \frac{x-2}{x^2-4} \approx 0.25$ (Actual limit is $\frac{1}{4}$.)

5.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} \approx 0.2887$ (Actual limit is $1/(2\sqrt{3})$.)

6.

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-0.2485	-0.2498	-0.2500	-0.2500	-0.2502	-0.2516

$\lim_{x\to -3} \frac{\sqrt{1-x}-2}{x+3} \approx -0.25$ (Actual limit is $-\frac{1}{4}$.)

7.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$\lim_{x\to 3} \frac{[1/(x+1)]-(1/4)}{x-3} \approx -0.0625$ (Actual limit is $-\frac{1}{16}$.)

8.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.0408	0.0401	0.0400	0.0400	0.0399	0.0392

$$\lim_{x \to 4} \frac{[x/(x+1)] - (4/5)}{x-4} \approx 0.04 \quad \left(\text{Actual limit is } \tfrac{1}{25}.\right)$$

9.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x \to 0} \frac{\sin x}{x} \approx 1.0000$$ (Actual limit is 1.) (Make sure you use radian mode.)

10.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	-0.0005	-0.0050	-0.0500

$$\lim_{x \to 0} \frac{\cos x - 1}{x} \approx 0.0000$$ (Actual limit is 0.)

11. $\lim_{x \to 3} (4 - x) = 1$

12. $\lim_{x \to 1} (x^2 + 2) = 3$

13. $\lim_{x \to 2} f(x) = \lim_{x \to 2} (4 - x) = 2$

14. $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 2) = 3$

15. $\lim_{x \to 5} \frac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x - 5|/(x - 5)$ equals -1, whereas for values of x to the right of 5, $|x - 5|/(x - 5)$ equals 1.

16. $\lim_{x \to 3} \frac{1}{x-3}$ does not exist since the function increases and decreases without bound as x approaches 3.

17. $\lim_{x \to \pi/2} \tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

18. $\lim_{x \to 0} \sec x = 1$

19. $\lim_{x \to 0} \cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0.

20. $\lim_{x \to 1} \sin(\pi x) = 0$

21. $\lim_{x \to 2} (3x + 2) = 8 = L$

$$|(3x + 2) - 8| < 0.01$$
$$|3x - 6| < 0.01$$
$$3|x - 2| < 0.01$$
$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

22. $\lim_{x \to 4} \left(4 - \frac{x}{2}\right) = 2$

$$\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$$
$$\left|2 - \frac{x}{2}\right| < 0.01$$
$$\left|-\frac{1}{2}(x - 4)\right| < 0.01$$
$$0 < |x - 4| < 0.02 = \delta$$

23. $\lim_{x\to 2}(x^2 - 3) = 1 = L$

$$|(x^2 - 3) - 1| < 0.01$$
$$|x^2 - 4| < 0.01$$
$$|(x+2)(x-2)| < 0.01$$
$$|x+2|\,|x-2| < 0.01$$
$$|x-2| < \frac{0.01}{|x+2|}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

24. $\lim_{x\to 5}(x^2 + 4) = 29$

$$|(x^2 + 4) - 29| < 0.01$$
$$|x^2 - 25| < 0.01$$
$$|(x+5)(x-5)| < 0.01$$
$$|x-5| < \frac{0.01}{|x+5|}$$

If we assume $4 < x < 6$, then $\delta = 0.01/11 \approx 0.0009$.

25. $\lim_{x\to 2}(x+3) = 5$

Given $\epsilon > 0$:

$$|(x+3) - 5| < \epsilon$$
$$|x-2| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

26. $\lim_{x\to -3}(2x+5) = -1$

Given $\epsilon > 0$:

$$|(2x+5) - (-1)| < \epsilon$$
$$|2x+6| < \epsilon$$
$$2|x+3| < \epsilon$$
$$|x+3| < \frac{\epsilon}{2} = \delta$$

Hence, let $\delta = \epsilon/2$.

27. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\epsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \epsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \epsilon$$
$$\tfrac{1}{2}|x - (-4)| < \epsilon$$
$$|x - (-4)| < 2\epsilon$$

Hence, let $\delta = 2\epsilon$.

28. $\lim_{x\to 1}\left(\frac{2}{3}x + 9\right) = \frac{2}{3}(1) + 9 = \frac{29}{3}$

Given $\epsilon > 0$:

$$\left|\left(\tfrac{2}{3}x + 9\right) - \tfrac{29}{3}\right| < \epsilon$$
$$\left|\tfrac{2}{3}x - \tfrac{2}{3}\right| < \epsilon$$
$$\tfrac{2}{3}|x - 1| < \epsilon$$
$$|x-1| < \tfrac{3}{2}\epsilon$$

Hence, let $\delta = (3/2)\epsilon$.

29. $\lim_{x\to 6} 3 = 3$

Given $\epsilon > 0$:

$$|3 - 3| < \epsilon$$
$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

30. $\lim_{x\to 2}(-1) = -1$

Given $\epsilon > 0$: $|-1 - (-1)| < \epsilon$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

31. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\epsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \epsilon$

$$\left|\sqrt[3]{x}\right| < \epsilon$$
$$|x| < \epsilon^3 = \delta$$

Hence, let $\delta = \epsilon^3$.

32. $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\epsilon > 0$: $\left|\sqrt{x} - 2\right| < \epsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \epsilon\left|\sqrt{x} + 2\right|$$
$$|x - 4| < \epsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\epsilon$. Then,

$$0 < |x-4| < \delta = 3\epsilon \Rightarrow |x-4| < \epsilon\left|\sqrt{x}+2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \epsilon.$$

33. $\lim_{x\to -2}|x-2| = |(-2) - 2| = 4$

Given $\epsilon > 0$:

$$\big||x-2| - 4\big| < \epsilon$$
$$|-(x-2) - 4| < \epsilon \quad (x - 2 < 0)$$
$$|-x-2| = |x+2| = |x - (-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

34. $\lim\limits_{x\to3} |x-3| = 0$

Given $\epsilon > 0$:

$$|(x-3)-0| < \epsilon$$
$$|x-3| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

35. $\lim\limits_{x\to1} (x^2+1) = 2$

Given $\epsilon > 0$:

$$|(x^2+1)-2| < \epsilon$$
$$|x^2-1| < \epsilon$$
$$|(x+1)(x-1)| < \epsilon$$
$$|x-1| < \frac{\epsilon}{|x+1|}$$

If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

36. $\lim\limits_{x\to-3} (x^2+3x) = 0$

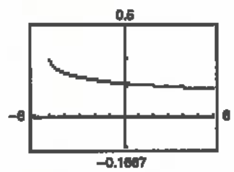
Given $\epsilon > 0$:

$$|(x^2+3x)-0| < \epsilon$$
$$|x(x+3)| < \epsilon$$
$$|x+3| < \frac{\epsilon}{|x|}$$

If we assume $-4 < x < -2$, then $\delta = \epsilon/4$.

37. $f(x) = \dfrac{\sqrt{x+5}-3}{x-4}$

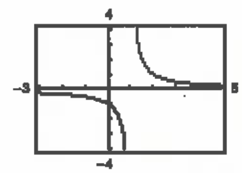
$\lim\limits_{x\to4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

38. $f(x) = \dfrac{x-3}{x^2-4x+3}$

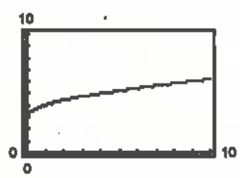
$\lim\limits_{x\to3} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$.

39. $f(x) = \dfrac{x-9}{\sqrt{x}-3}$

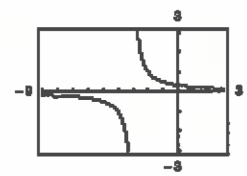
$\lim\limits_{x\to9} f(x) = 6$



The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

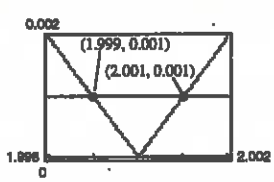
40. $f(x) = \dfrac{x-3}{x^2-9}$

$\lim\limits_{x\to3} f(x) = \dfrac{1}{6}$



The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \frac{1}{6}\right)$.

41.



Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x-2| < 0.001, \left|\frac{x^2-4}{x-2} - 4\right| < 0.001.$$

42. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim\limits_{x\to2} f(x) = 4$ has no bearing on the value of f at 2.

43. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$. From Exercise 9, we have

$$\lim_{x\to0} \frac{\sin x}{x} = 1.$$

44. True

45. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$f(4) = 10$

$\lim_{x \to 4} f(x) = \lim_{x \to 4} (x^2 - 4x) = 0 \neq 10$

46. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$\lim_{x \to 4} f(x) = \lim_{x \to 4} (x^2 - 4x) = 0$ and $f(4) = 10 \neq 0$

47. Answers will vary.

48. $\lim_{x \to 4} \dfrac{x^2 - x - 12}{x - 4} = 7$

n	$4 + [0.1]^n$	$f(4 + [0.1]^n)$
1	4.1	7.1
2	4.01	7.01
3	4.001	7.001
4	4.0001	7.0001

n	$4 - [0.1]^n$	$f(4 - [0.1]^n)$
1	3.9	6.9
2	3.99	6.99
3	3.999	6.999
4	3.9999	6.9999

49. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

50. $f(x) = mx + b,\ m \neq 0$

$\lim_{x \to c} (mx + b) = mc + b$

$$|(mx + b) - (mc + b)| < \epsilon$$

$$|m(x - c)| < \epsilon$$

$$|x - c| < \frac{\epsilon}{|m|} = \delta$$

Hence for $\epsilon > 0$, let $\delta = \epsilon/|m|$.

51. $\lim_{x \to c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \epsilon.$$

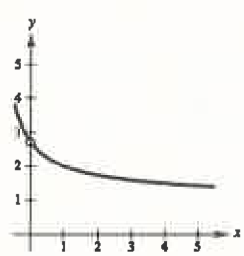
This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x \to c} f(x) = L$.

52. $f(x) = (1 + x)^{1/x}$

$\lim_{x \to 0} (1 + x)^{1/x} = e \approx 2.71828$

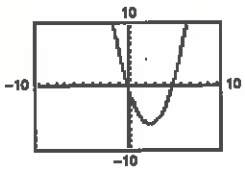


x	$f(x)$
-0.1	2.867972
-0.01	2.731999
-0.001	2.719642
-0.0001	2.718418
-0.00001	2.718295
-0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

Section 1.3 Evaluating Limits Analytically

1.

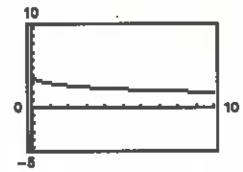


$h(x) = x^2 - 5x$

(a) $\lim_{x \to 5} h(x) = 0$

(b) $\lim_{x \to -1} h(x) = 6$

2.

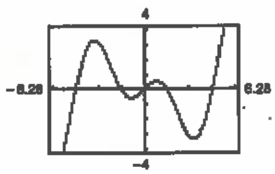


$g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$

(a) $\lim_{x \to 4} g(x) = 2.4$

(b) $\lim_{x \to 0} g(x) = 4$

3.



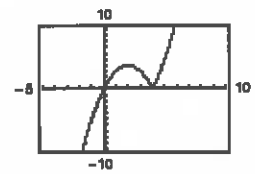
$f(x) = x \cos x$

(a) $\lim_{x \to 0} f(x) = 0$

(b) $\lim_{x \to \pi/3} f(x) \approx 0.524$

$\left(= \dfrac{\pi}{6}\right)$

4.



$f(t) = t|t - 4|$

(a) $\lim_{t \to 4} f(t) = 0$

(b) $\lim_{t \to -1} f(t) = -5$

5. $\lim_{x \to 4} x^2 = 4^2 = 16$

6. $\lim_{x \to -3} (3x + 2) = 3(-3) + 2 = 7$

7. $\lim_{x \to 0} (2x - 1) = 2(0) - 1 = -1$

8. $\lim_{x \to 1} (-x^2 + 1) = -(1)^2 + 1 = 0$

9. $\lim_{x \to 2} (-x^2 + x - 2) = -(2)^2 + (2) - 2 = -4$

10. $\lim_{x \to 1} (3x^3 - 2x^2 + 4) = 3(1)^3 - 2(1)^2 + 4 = 5$

11. $\lim_{x \to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

12. $\lim_{x \to 4} \sqrt[3]{x + 4} = \sqrt[3]{4 + 4} = 2$

13. $\lim_{x \to -4} (x + 3)^2 = (-4 + 3)^2 = 1$

14. $\lim_{x \to 0} (2x - 1)^3 = [2(0) - 1]^3 = -1$

15. $\lim_{x \to 2} \dfrac{1}{x} = \dfrac{1}{2}$

16. $\lim_{x \to -3} \dfrac{2}{x + 2} = \dfrac{2}{-3 + 2} = -2$

17. $\lim_{x \to -1} \dfrac{x^2 + 1}{x} = \dfrac{(-1)^2 + 1}{-1} = -2$

18. $\lim_{x \to 3} \dfrac{\sqrt{x + 1}}{x - 4} = \dfrac{\sqrt{3 + 1}}{3 - 4} = -2$

19. $\lim_{x \to \pi/2} \sin x = \sin \dfrac{\pi}{2} = 1$

20. $\lim_{x \to \pi} \tan x = \tan \pi = 0$

21. $\lim_{x \to 1} \cos \pi x = \cos \pi = -1$

22. $\lim_{x \to 1} \sin \dfrac{\pi x}{2} = \sin \dfrac{\pi}{2} = 1$

23. $\lim_{x \to 0} \sec 2x = \sec 0 = 1$

24. $\lim_{x \to \pi} \cos 3x = \cos 3\pi = -1$

25. $\lim_{x \to 5\pi/6} \sin x = \sin \dfrac{5\pi}{6} = \dfrac{1}{2}$

26. $\lim_{x \to 5\pi/3} \cos x = \cos \dfrac{5\pi}{3} = \dfrac{1}{2}$

27. $\lim_{x \to 3} \tan\left(\dfrac{\pi x}{4}\right) = \tan \dfrac{3\pi}{4} = -1$

28. $\lim_{x \to 7} \sec\left(\dfrac{\pi x}{6}\right) = \sec \dfrac{7\pi}{6} = \dfrac{-2\sqrt{3}}{3}$

29. (a) $\lim_{x\to c} [5g(x)] = 5 \lim_{x\to c} g(x) = 5(3) = 15$

(b) $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + 3 = 5$

(c) $\lim_{x\to c} [f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (2)(3) = 6$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{2}{3}$

30. (a) $\lim_{x\to c} [4f(x)] = 4 \lim_{x\to c} f(x) = 4\left(\dfrac{3}{2}\right) = 6$

(b) $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = \dfrac{3}{2} + \dfrac{1}{2} = 2$

(c) $\lim_{x\to c} [f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = \left(\dfrac{3}{2}\right)\left(\dfrac{1}{2}\right) = \dfrac{3}{4}$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{3/2}{1/2} = 3$

31. (a) $\lim_{x\to c} [f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c} [3f(x)] = 3 \lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c} [f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

32. (a) $\lim_{x\to c} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c} \dfrac{f(x)}{18} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \dfrac{27}{18} = \dfrac{3}{2}$

(c) $\lim_{x\to c} [f(x)]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c} [f(x)]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

33. $f(x) = -2x + 1$ and $g(x) = \dfrac{-2x^2 + x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = 1$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 3$

34. $f(x) = x - 3$ and $h(x) = \dfrac{x^2 - 3x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to -2} h(x) = \lim_{x\to -2} f(x) = -5$

(b) $\lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = -3$

35. $f(x) = x(x + 1)$ and $g(x) = \dfrac{x^3 - x}{x - 1}$ agree except at $x = 1$.

(a) $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 0$

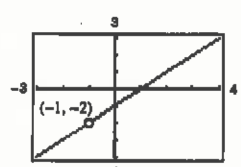
36. $g(x) = \dfrac{1}{x - 1}$ and $f(x) = \dfrac{x}{x^2 - x}$ agree except at $x = 0$.

(a) $\lim_{x\to 1} f(x)$ does not exist.

(b) $\lim_{x\to 0} f(x) = -1$

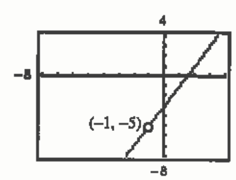
37. $f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -2$



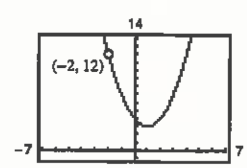
38. $f(x) = \dfrac{2x^2 - x - 3}{x + 1}$ and $g(x) = 2x - 3$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -5$



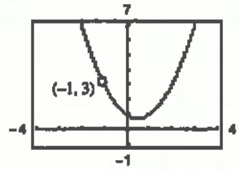
39. $f(x) = \dfrac{x^3 + 8}{x + 2}$ and $g(x) = x^2 - 2x + 4$ agree except at $x = -2$.

$\lim_{x\to -2} f(x) = \lim_{x\to -2} g(x) = 12$



40. $f(x) = \dfrac{x^3+1}{x+1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = 3$$



41. $$\lim_{x\to 5}\frac{x-5}{x^2-25} = \lim_{x\to 5}\frac{x-5}{(x+5)(x-5)} = \lim_{x\to 5}\frac{1}{x+5} = \frac{1}{10}$$

42. $$\lim_{x\to 2}\frac{2-x}{x^2-4} = \lim_{x\to 2}\frac{-(x-2)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{-1}{x+2} = -\frac{1}{4}$$

43. $$\lim_{x\to 1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to 1}\frac{(x+2)(x-1)}{(x+1)(x-1)} = \lim_{x\to 1}\frac{x+2}{x+1} = \frac{3}{2}$$

44. $$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}\cdot\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$$
$$= \lim_{x\to 0}\frac{2+x-2}{\left(\sqrt{2+x}+\sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

45. $$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x} = \lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}$$
$$= \lim_{x\to 0}\frac{3+x-3}{\left(\sqrt{3+x}+\sqrt{3}\right)x} = \lim_{x\to 0}\frac{1}{\left(\sqrt{3+x}+\sqrt{3}\right)} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

46. $$\lim_{x\to 0}\frac{\frac{1}{x+4}-\frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{4-(x+4)}{4(x+4)}}{x} = \lim_{x\to 0}\frac{-1}{4(x+4)} = -\frac{1}{16}$$

47. $$\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0}\frac{\frac{2-(2+x)}{2(2+x)}}{x} = \lim_{x\to 0}\frac{-1}{2(2+x)} = -\frac{1}{4}$$

48. $$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3}\frac{x-3}{(x-3)\left[\sqrt{x+1}+2\right]} = \lim_{x\to 3}\frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$$

49. $$\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{2x+2\Delta x-2x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$$

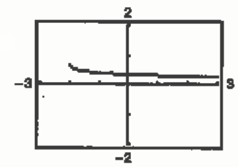
50. $$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x+\Delta x) = 2x$$

51. $$\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$$
$$= \lim_{\Delta x\to 0}(2x+\Delta x-2) = 2x-2$$

52. $$\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

53. $$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354$$

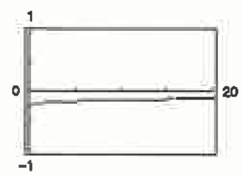


x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.345	?	0.354	0.353	0.349

Analytically, $$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}}$$

$$= \lim_{x \to 0} \frac{x + 2 - 2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

54. $$f(x) = \frac{4 - \sqrt{x}}{x - 16}$$



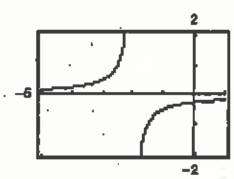
x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	$-.1252$	$-.125$	$-.125$	?	$-.125$	$-.125$	$-.1248$

It appears that the limit is -0.125.

Analytically, $$\lim_{x \to 16} \frac{4 - \sqrt{x}}{x - 16} = \lim_{x \to 16} \frac{(4 - \sqrt{x})}{(\sqrt{x} + 4)(\sqrt{x} - 4)}$$

$$= \lim_{x \to 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.$$

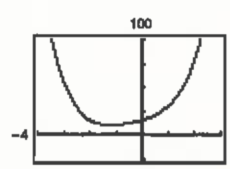
55. $$\lim_{x \to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = -\frac{1}{4}$$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238

Analytically, $$\lim_{x \to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \to 0} \frac{2 - (2 + x)}{2(2 + x)} \cdot \frac{1}{x} = \lim_{x \to 0} \frac{-x}{2(2 + x)} \cdot \frac{1}{x} = \lim_{x \to 0} \frac{-1}{2(2 + x)} = -\frac{1}{4}.$$

56. $$\lim_{x \to 2} \frac{x^5 - 32}{x - 2} = 80$$



x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

Analytically, $$\lim_{x \to 2} \frac{x^5 - 32}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2}$$

$$= \lim_{x \to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$$

(*Hint:* Use long division to factor $x^5 - 32$.)

57. $\lim\limits_{x\to 0}\dfrac{\sin x}{5x} = \lim\limits_{x\to 0}\left[\left(\dfrac{\sin x}{x}\right)\left(\dfrac{1}{5}\right)\right] = (1)\left(\dfrac{1}{5}\right) = \dfrac{1}{5}$

58. $\lim\limits_{x\to 0}\dfrac{3(1-\cos x)}{x} = \lim\limits_{x\to 0}\left[3\left(\dfrac{1-\cos x}{x}\right)\right] = (3)(0) = 0$

59. $\lim\limits_{\theta\to 0}\dfrac{\sec\theta - 1}{\theta\sec\theta} = \lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta} = 0$

60. $\lim\limits_{\theta\to 0}\dfrac{\cos\theta\tan\theta}{\theta} = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$

61. $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x} = \lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

62. $\lim\limits_{x\to 0}\dfrac{\tan^2 x}{x} = \lim\limits_{x\to 0}\dfrac{\sin^2 x}{x\cos^2 x} = \lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\cdot\dfrac{\sin x}{\cos^2 x}\right]$

$= (1)(0) = 0$

63. $\lim\limits_{h\to 0}\dfrac{(1-\cos h)^2}{h} = \lim\limits_{h\to 0}\left[\dfrac{1-\cos h}{h}(1-\cos h)\right]$

$= (0)(0) = 0$

64. $\lim\limits_{\phi\to\pi}\phi\sec\phi = \pi(-1) = -\pi$

65. $\lim\limits_{x\to\pi/2}\dfrac{\cos x}{\cot x} = \lim\limits_{x\to\pi/2}\sin x = 1$

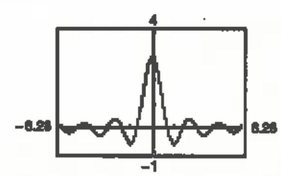
66.
$$\begin{aligned}\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} &= \lim_{x\to\pi/4}\frac{\cos x-\sin x}{\sin x\cos x-\cos^2 x}\\ &= \lim_{x\to\pi/4}\frac{-(\sin x-\cos x)}{\cos x(\sin x-\cos x)}\\ &= \lim_{x\to\pi/4}\frac{-1}{\cos x}\\ &= \lim_{x\to\pi/4}(-\sec x)\\ &= -\sqrt{2}\end{aligned}$$

67. $\lim\limits_{t\to 0}\dfrac{\sin^2 t}{t^2} = \lim\limits_{t\to 0}\left(\dfrac{\sin t}{t}\right)^2 = (1)^2 = 1$

68. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0}\left[2\left(\dfrac{\sin 2x}{2x}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{3x}{\sin 3x}\right)\right] = 2(1)\left(\dfrac{1}{3}\right)(1) = \dfrac{2}{3}$

69. $f(t) = \dfrac{\sin 3t}{t}$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

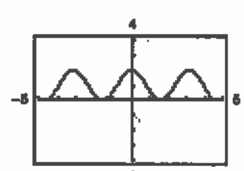


The limit appear to equal 3.

Analytically, $\lim\limits_{t\to 0}\dfrac{\sin 3t}{t} = \lim\limits_{t\to 0}3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

70. $f(h) = (1 + \cos 2h)$

h	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(h)$	1.98	1.9998	2	?	2	1.9998	1.98

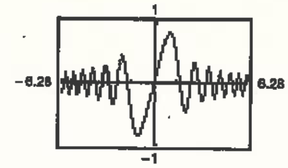


The limit appear to equal 2.

Analytically, $\lim\limits_{h\to 0}(1+\cos 2h) = 1 + \cos(0) = 1 + 1 = 2.$

71. $f(x) = \dfrac{\sin x^2}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998

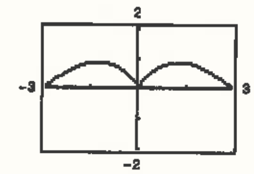


The limit appear to equal 0.

Analytically, $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x} = \lim\limits_{x\to 0} x\left(\dfrac{\sin x^2}{x^2}\right) = 0(1) = 0.$

72. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215



The limit appear to equal 0.

Analytically, $\lim\limits_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}} = \lim\limits_{x\to 0} \sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

73. $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{2(x+h) + 3 - (2x+3)}{h} = \lim\limits_{h\to 0} \dfrac{2x + 2h + 3 - 2x - 3}{h} = \lim\limits_{h\to 0} \dfrac{2h}{h} = 2$

74. $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$$= \lim_{h\to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

75. $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{4}{x+h} - \dfrac{4}{x}}{h} = \lim\limits_{h\to 0} \dfrac{4x - 4(x+h)}{(x+h)xh} = \lim\limits_{h\to 0} \dfrac{-4}{(x+h)x} = \dfrac{-4}{x^2}$

76. $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h} = \lim\limits_{h\to 0} \dfrac{x^2 + 2xh + h^2 - 4x - 4h - x^2 + 4x}{h}$

$$= \lim_{h\to 0} \frac{h(2x + h - 4)}{h} = \lim_{h\to 0} (2x + h - 4) = 2x - 4$$

77. $\lim\limits_{x\to 0} (4 - x^2) \le \lim\limits_{x\to 0} f(x) \le \lim\limits_{x\to 0} (4 + x^2)$

$$4 \le \lim_{x\to 0} f(x) \le 4$$

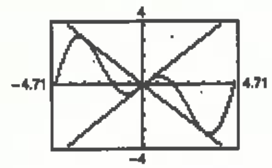
Therefore, $\lim\limits_{x\to 0} f(x) = 4.$

78. $\lim\limits_{x\to a} [b - |x - a|] \le \lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} [b + |x - a|]$

$$b \le \lim_{x\to a} f(x) \le b$$

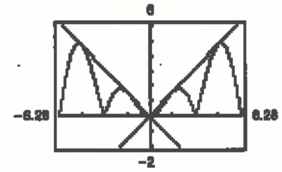
Therefore, $\lim\limits_{x\to a} f(x) = b.$

79. $f(x) = x\cos x$



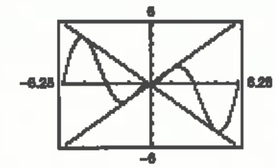
$\lim_{x\to 0}(x\cos x) = 0$

80. $f(x) = |x\sin x|$



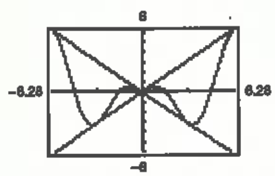
$\lim_{x\to 0}|x\sin x| = 0$

81. $f(x) = |x|\sin x$



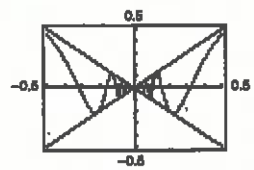
$\lim_{x\to 0}|x|\sin x = 0$

82. $f(x) = |x|\cos x$



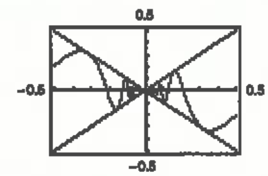
$\lim_{x\to 0}|x|\cos x = 0$

83. $f(x) = x\sin\dfrac{1}{x}$



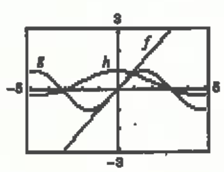
$\lim_{x\to 0}\left(x\sin\dfrac{1}{x}\right) = 0$

84. $f(x) = x\cos\dfrac{1}{x}$



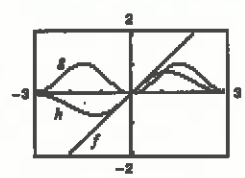
$\lim_{x\to 0}\left(x\cos\dfrac{1}{x}\right) = 0$

85. $f(x) = x$, $g(x) = \sin x$, $h(x) = \dfrac{\sin x}{x}$



When you are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. Thus, $|g|/|f| \approx 1$ when x is "close to" 0.

86. $f(x) = x$, $g(x) = \sin^2 x$, $h(x) = \dfrac{\sin^2 x}{x}$



When you are "close to" 0 the magnitude of g is "smaller" than the magnitude of f and the magnitude of g is approaching zero "faster" than the magnitude of f. Thus, $|g|/|f| \approx 0$ when x is "close to" 0.

87. $s(t) = -16t^2 + 1000$

$$\lim_{t\to 5}\frac{s(5)-s(t)}{5-t} = \lim_{t\to 5}\frac{600-(-16t^2+1000)}{5-t} = \lim_{t\to 5}\frac{16(t+5)(t-5)}{-(t-5)} = \lim_{t\to 5} -16(t+5) = -160 \text{ ft/sec}$$

88. $s(t) = -16t^2 + 1000 = 0$ when $t = \sqrt{\dfrac{1000}{16}} = \dfrac{5\sqrt{10}}{2}$ seconds

$$\lim_{t\to 5\sqrt{10}/2}\frac{s\left(\frac{5\sqrt{10}}{2}\right)-s(t)}{\frac{5\sqrt{10}}{2}-t} = \lim_{t\to 5\sqrt{10}/2}\frac{0-(-16t^2+1000)}{\frac{5\sqrt{10}}{2}-t}$$

$$= \lim_{t\to 5\sqrt{10}/2}\frac{16\left(t^2-\frac{125}{2}\right)}{\frac{5\sqrt{10}}{2}-t} = \lim_{t\to 5\sqrt{10}/2}\frac{16\left(t+\frac{5\sqrt{10}}{2}\right)\left(t-\frac{5\sqrt{10}}{2}\right)}{-\left(t-\frac{5\sqrt{10}}{2}\right)}$$

$$= \lim_{t\to 5\sqrt{10}/2} -16\left(t+\frac{5\sqrt{10}}{2}\right) = -80\sqrt{10} \text{ ft/sec} \approx -253 \text{ ft/sec}$$

89. $s(t) = -4.9t^2 + 150$

$$\lim_{t\to3}\frac{s(3)-s(t)}{3-t} = \lim_{t\to3}\frac{-4.9(3^2)+150-(-4.9t^2+150)}{3-t} = \lim_{t\to3}\frac{-4.9(9-t^2)}{3-t}$$

$$= \lim_{x\to3}\frac{-4.9(3-t)(3+t)}{3-t} = \lim_{x\to3} -4.9(3+t) = -29.4 \text{ m/sec}$$

90. $-4.9t^2 + 150 = 0$ when $t = \sqrt{\dfrac{150}{4.9}} = \sqrt{\dfrac{1500}{49}} \approx 5.53$ seconds.

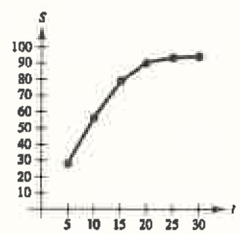
The velocity at time $t = a$ is

$$\lim_{t\to a}\frac{s(a)-s(t)}{a-t} = \lim_{t\to a}\frac{(-4.9a^2+150)-(-4.9t^2+150)}{a-t} = \lim_{t\to a}\frac{-4.9(a-t)(a+t)}{a-t}$$

$$= \lim_{t\to a} -4.9(a+t) = -2a(4.9) = -9.8a \text{ m/sec.}$$

Hence, if $a = \sqrt{1500/49}$, the velocity is $-9.8\sqrt{1500/49} \approx -54.2$ m/sec.

91. (a)



(b) Yes. As time increases, the typing speed seems to approach 95 or 100. The rate of increase is decreasing.

92. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ do not exist.

$$\lim_{x\to0}[f(x)+g(x)] = \lim_{x\to0}\left[\frac{1}{x}+\left(-\frac{1}{x}\right)\right] = \lim_{x\to0}[0] = 0$$

93. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, since $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}[f(x)+g(x)]$, which is a contradiction. Hence, $\lim_{x\to c} g(x)$ does not exist.

94. Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

95. Given $f(x) = x^n$, n is a positive integer, then

$$\lim_{x\to c} x^n = \lim_{x\to c}(xx^{n-1}) = \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right]$$

$$= c\left[\lim_{x\to c}(xx^{n-2})\right] = c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right]$$

$$= c(c)\lim_{x\to c}(xx^{n-3}) = \cdots = c^n.$$

96. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x\to c}[bf(x)] = bL$.

97. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$. Then $f(x) < g(x)$ for all $x \neq 0$. But $\lim_{x\to0} f(x) = \lim_{x\to0} g(x) = 0$.

98. Given $\lim_{x\to c} f(x) = 0$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$. Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \epsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x\to c} |f(x)| = 0$.

99.
$$-M|f(x)| \le f(x)g(x) \le M|f(x)|$$
$$\lim_{x\to c}(-M|f(x)|) \le \lim_{x\to c} f(x)g(x) \le \lim_{x\to c}(M|f(x)|)$$
$$-M(0) \le \lim_{x\to c} f(x)g(x) \le M(0)$$
$$0 \le \lim_{x\to c} f(x)g(x) \le 0$$

Therefore, $\lim_{x\to c} f(x)g(x) = 0$.

100. If $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} [-|f(x)|] = 0$.
$$-|f(x)| \le f(x) \le |f(x)|$$
$$\lim_{x\to c} [-|f(x)|] \le \lim_{x\to c} f(x) \le \lim_{x\to c} |f(x)|$$
$$0 \le \lim_{x\to c} f(x) \le 0$$

Therefore, $\lim_{x\to c} f(x) = 0$.

101. Given $\lim_{x\to c} f(x) = L$:

For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.
Since $||f(x)| - |L|| \le |f(x) - L| < \epsilon$ for $|x - c| < \delta$, then $\lim_{x\to c} |f(x)| = |L|$.

102. Let
$$f(x) = \begin{cases} 4, & \text{if } x \ge 0 \\ -4, & \text{if } x < 0 \end{cases}$$
$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4.$$

$\lim_{x\to 0} f(x)$ does not exist since for $x < 0, f(x) = -4$ and for $x \ge 0, f(x) = 4$.

103.
$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$$
$$= \lim_{x\to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1 + \cos x)}$$
$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x}$$
$$= \left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{x\to 0} \frac{\sin x}{1 + \cos x}\right]$$
$$= (1)(0) = 0$$

104. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

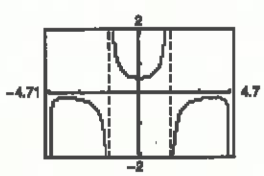
$\lim_{x\to 0} g(x) = 0$.

When x is "close to" 0, both parts of the function are "close to" 0.

105. $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \ne 0, \pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}$$
$$= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

Hence, $\displaystyle\lim_{x\to 0} \frac{\sec x - 1}{x^2} = \lim_{x\to 0} \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$
$$= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

106. (a) $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}$

$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$

$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$

$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$

(b) Thus, $\frac{1-\cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1-\cos x \approx \frac{1}{2}x^2$

$\Rightarrow \cos x \approx 1 - \frac{1}{2}x^2$ for $x \approx 0$.

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

107. Two functions agree at all but one point means that the functions are identical for all x in their domain, except possibly at one value of x.

Section 1.4 Continuity and One-Sided Limits

1. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

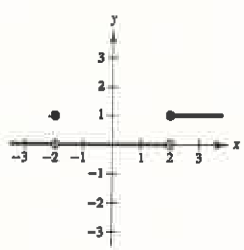
(d) The limit does not exist at $x = c$.

2. A discontinuity at $x = c$ is removable if you can define (or redefine) the function at $x = c$ in such a way that the new function is continuous at $x = c$.

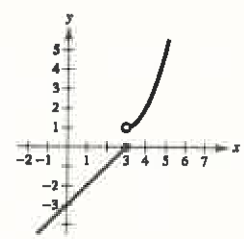
(a) $f(x) = \frac{|x-2|}{x-2}$

(b) $f(x) = \frac{\sin(x+2)}{x+2}$

(c) $f(x) = \begin{cases} 1, & \text{if } x \geq 2 \\ 0, & \text{if } -2 < x < 2 \\ 1, & \text{if } x = -2 \\ 0, & \text{if } x < -2 \end{cases}$



3.



The function is not continuous at $x = 3$ because $\lim_{x\to 3^+} f(x) = 1 \neq 0 = \lim_{x\to 3^-} f(x)$.

4. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

5. (a) $\lim_{x\to 3^+} f(x) = 1$

(b) $\lim_{x\to 3^-} f(x) = 1$

(c) $\lim_{x\to 3} f(x) = 1$

6. (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

7. (a) $\lim_{x\to 3^+} f(x) = 0$

(b) $\lim_{x\to 3^-} f(x) = 0$

(c) $\lim_{x\to 3} f(x) = 0$

8. (a) $\lim_{x\to -2^+} f(x) = 2$

(b) $\lim_{x\to -2^-} f(x) = 2$

(c) $\lim_{x\to -2} f(x) = 2$

9. (a) $\lim_{x\to 3^+} f(x) = 3$

(b) $\lim_{x\to 3^-} f(x) = -3$

(c) $\lim_{x\to 3} f(x)$ does not exist.

10. (a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to -1^-} f(x) = 2$

(c) $\lim_{x\to -1} f(x)$ does not exist.

11. $\lim_{x\to 5^+} \frac{x-5}{x^2-25} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$

12. $\lim_{x\to 2^+} \frac{2-x}{x^2-4} = \lim_{x\to 2^+} -\frac{1}{x+2} = -\frac{1}{4}$

13. $\lim_{x\to 2^+} \frac{x}{\sqrt{x^2-4}}$ does not exist since $\frac{x}{\sqrt{x^2-4}}$ grows without bound as $x \to 2^+$.

14. $\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$

$= \lim_{x\to 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)}$

$= \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$

15. $\lim_{x\to 0} \frac{|x|}{x}$ does not exist since $\lim_{x\to 0^+} \frac{|x|}{x} = 1$ and

$\lim_{x\to 0^-} \frac{|x|}{x} = -1.$

16. $\lim_{x\to 2^+} \frac{|x-2|}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1$

$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$

$\lim_{x\to 2^+} \frac{|x-2|}{x-2} \neq \lim_{x\to 2^-} \frac{|x-2|}{x-2}$

Thus, the limit does not exist.

17. $\lim_{\Delta x\to 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$

$= \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)}$

$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$

18. $\lim_{\Delta x\to 0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$

$= \lim_{\Delta x\to 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0^+} (2x + \Delta x + 1)$

$= 2x + 0 + 1 = 2x + 1$

19. $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{12-2x}{3} = 2$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+2}{2} = \frac{5}{2}$

$\lim_{x\to 3} f(x)$ does not exist.

20. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-x^2 + 4x - 2) = 2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 4x + 6) = 2$

$\lim_{x\to 2} f(x) = 2$

21. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3+1) = 2$

$\lim_{x\to 1} f(x) = 2$

22. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x) = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1$

$\lim_{x\to 1} f(x)$ does not exist.

23. $\lim_{x\to\pi} \cot x$ does not exist since

$\lim_{x\to\pi^+} \cot x$ and $\lim_{x\to\pi^-} \cot x$ do not exist.

24. $\lim_{x\to\pi/2} \sec x$ does not exist since

$\lim_{x\to(\pi/2)^+} \sec x$ and $\lim_{x\to(\pi/2)^-} \sec x$ do not exist.

25. $\lim_{x\to 3^-} (2[\![x]\!] - 1) = 2(2) - 1 = 3$

$([\![x]\!] = 2$ for $2 < x < 3)$

26. $\lim_{x\to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

27. $f(x) = \dfrac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

28. $f(x) = \dfrac{x^2 - 1}{x + 1}$

has a discontinuity at $x = -1$ since $f(-1)$ is not defined.

29. $f(x) = \dfrac{[\![x]\!]}{2} + x$

has discontinuities at each integer k since $\lim_{x\to k^-} f(x) \neq \lim_{x\to k^+} f(x)$.

30. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has discontinuity at $x = 1$ since $f(1) = 2 \neq \lim_{x\to 1} f(x) = 1$.

31. $f(x) = x^2 - 2x + 1$ is continuous for all real x.

32. $f(x) = \dfrac{1}{x^2 + 1}$ is continuous for all real x.

33. $f(x) = x + \sin x$ is continuous for all real x.

34. $f(x) = \cos \dfrac{\pi x}{2}$ is continuous for all real x.

35. $f(x) = \dfrac{1}{x - 1}$ has a nonremovable discontinuity at $x = 1$ since $\lim_{x\to 1} f(x)$ does not exist.

36. $f(x) = \dfrac{x}{x^2 - 1}$ has nonremovable discontinuities at $x = 1$ and $x = -1$ since $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$ do not exist.

37. $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real x.

38. $f(x) = \dfrac{x - 3}{x^2 - 9}$ has a nonremovable discontinuity at $x = -3$ since $\lim_{x\to -3} f(x)$ does not exist, and has a removable discontinuity at $x = 3$ since

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{1}{x + 3} = \frac{1}{6}.$$

39. $f(x) = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x\to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

40. $f(x) = \dfrac{x - 1}{(x + 2)(x - 1)}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist, and has a removable discontinuity at $x = 1$ since

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x + 2} = \frac{1}{3}.$$

41. $f(x) = \dfrac{|x + 2|}{x + 2}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist.

42. $f(x) = \dfrac{|x - 3|}{x - 3}$

has a nonremovable discontinuity at $x = 3$ since $\lim_{x\to 3} f(x)$ does not exist.

43. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a possible discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

3. $f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

44. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a possible discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (-2x + 3) = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

3. $f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

45. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$ has a possible discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(\dfrac{x}{2} + 1\right) = 2 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (3 - x) = 1 \end{array}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

46. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ has a possible discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (-2x) = -4 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^2 - 4x + 1) = -3 \end{array}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

47. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases} = \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$ has possible discontinuities at $x = 1, x = 5$.

1. $f(1) = \csc \dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc \dfrac{5\pi}{6} = 2$

2. $\lim\limits_{x\to 1} f(x) = 2$ $\qquad \lim\limits_{x\to 5} f(x) = 2$

3. $f(1) = \lim\limits_{x\to 1} f(x)$ $\qquad f(5) = \lim\limits_{x\to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

48. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases} = \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$ has possible discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $\qquad f(1) = 1$

2. $\lim\limits_{x\to -1} f(x) = -1$ $\qquad \lim\limits_{x\to 1} f(x) = 1$

3. $f(-1) = \lim\limits_{x\to -1} f(x)$ $\qquad f(1) = \lim\limits_{x\to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

49. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

50. $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

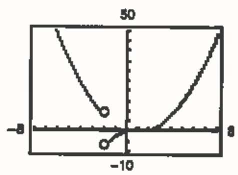
51. $f(x) = [\![x - 1]\!]$ has nonremovable discontinuities at each integer k.

52. $f(x) = x - [\![x]\!]$ has nonremovable discontinuities at each integer k.

53. $\lim\limits_{x\to 0^+} f(x) = 0$

$\lim\limits_{x\to 0^-} f(x) = 0$

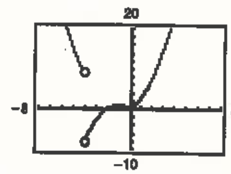
f is not continuous at $x = -2$.



54. $\lim\limits_{x\to 0^+} f(x) = 0$

$\lim\limits_{x\to 0^-} f(x) = 0$

f is not continuous at $x = -4$



55. $f(2) = 8$

Find a so that $\lim\limits_{x\to 2^+} ax^2 = 8 \implies a = \dfrac{8}{2^2} = 2$.

56. $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} \dfrac{4 \sin x}{x} = 4$

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} (a - 2x) = a$

Let $a = 4$.

57. Find a and b such that $\lim\limits_{x\to -1^+} (ax + b) = -a + b = 2$ and $\lim\limits_{x\to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)\ 3a + b &= -2 \\ \hline 4a \qquad &= -4 \\ a &= -1 \\ b &= \ 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

58. $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim\limits_{x\to a} (x + a) = 2a$

Find a such that $2a = 8 \implies a = 4$.

59. $f(g(x)) = (x - 1)^2$

Continuous for all real x.

60. $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$. Continuous for all $x > 1$.
Because $f \circ g$ is not defined for $x < 1$, it is better to say that $f \circ g$ is discontinuous from the right at $x = 1$.

61. $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

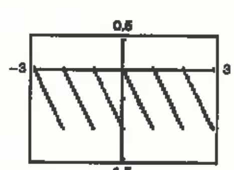
Nonremovable discontinuities at $x = \pm 1$

62. $f(g(x)) = \sin x^2$

Continuous for all real x

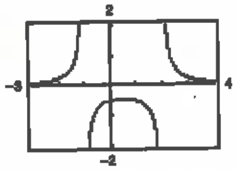
63. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



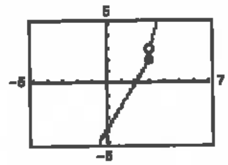
64. $h(x) = \dfrac{1}{(x+1)(x-2)}$

Nonremovable discontinuity at $x = -1$ and $x = 2$.

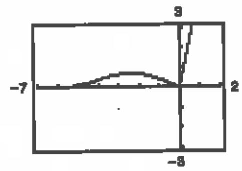


65. $f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



66. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$



$f(0) = 5(0) = 0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (5x) = 0$

Therefore, $\lim\limits_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)

67. $f(x) = \dfrac{x}{x^2 + 1}$

Continuous on $(-\infty, \infty)$

68. $f(x) = x\sqrt{x+3}$

Continuous on $[-3, \infty]$

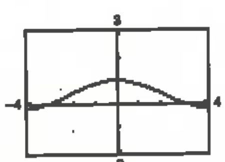
69. $f(x) = \csc \dfrac{x}{2}$

Continuous on: $\ldots, (-2\pi, 0), (0, 2\pi), (2\pi, 4\pi), \ldots$

70. $f(x) = \dfrac{x+1}{\sqrt{x}}$

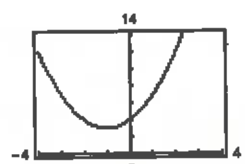
Continuous on $(0, \infty)$

71. $f(x) = \dfrac{\sin x}{x}$



The graph appears to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

72. $f(x) = \dfrac{x^3 - 8}{x - 2}$



The graph appears to be continuous on the interval $[-4, 4]$. Since $f(2)$ is not defined, we know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

73. $f(x) = x^2 - 4x + 3$

$f(x)$ is continuous on $[2, 4]$.

$f(2) = -1$ and $f(4) = 3$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 2 and 4.

74. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1.

75. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

76. $f(x) = x^3 + 3x - 2$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -2$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.5961$.

77. $g(t) = 2\cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

78. $h(\theta) = 1 + \theta - 3\tan\theta$

h is continuous on $[0, 1]$.

$h(0) = 1 > 0$ and $h(1) \approx -2.67 < 0$.

By the Intermediate Value Theorem, $h(\theta) = 0$ for at least one value θ between 0 and 1. Using a graphing utility, we find that $\theta \approx 0.4503$.

79. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$$-1 < 11 < 29$$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$
$$c = 3 \ (x = -4 \text{ is not in the interval.})$$

Thus, $f(3) = 11$.

80. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$$-1 < 0 < 8$$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$
$$c = 2 \ (x = 4 \text{ is not in the interval.})$$

Thus, $f(2) = 0$.

81. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$$-2 < 4 < 19$$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$
$$(x^2 + x + 3 \text{ has no real solution.})$$
$$c = 2.$$

Thus, $f(2) = 4$.

82. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$$f\left(\frac{5}{2}\right) = \frac{35}{6} \text{ and } f(4) = \frac{20}{3}$$
$$\frac{35}{6} < 6 < \frac{20}{3}$$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$
$$c = 3 \ (x = 2 \text{ is not in the interval.})$$

Thus, $f(3) = 6$.

83. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r.

$V(1) = \frac{4}{3}\pi \approx 4.19$

$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$

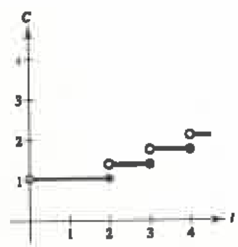
Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

84. $C = \begin{cases} 1.04, & 0 < t \le 2 \\ 1.04 + 0.36[\![t - 1]\!], & t > 2, t \text{ is not an integer} \\ 1.04 + 0.36(t - 2), & t > 2, t \text{ is an integer} \end{cases}$

Nonremovable discontinuity at each integer greater than 2.

You can also write C as

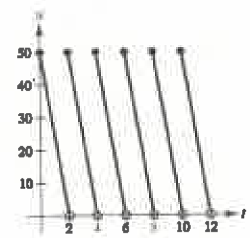
$$C = \begin{cases} 1.04, & 0 < t \le 2 \\ 1.04 - 0.36[\![2 - t]\!], & t > 2 \end{cases}$$



85. $N(t) = 25\left(2\left[\!\!\left[\frac{t+2}{2}\right]\!\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



86. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

Since $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

87. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

88. Let c be any real number. Then $\lim_{x \to c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

89. If $x = 0$, then $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$. Hence, f is continuous at $x = 0$.

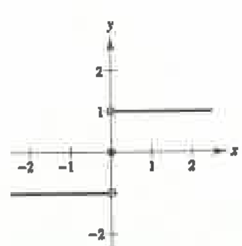
If $x \ne 0$, then $\lim_{t \to x} f(t) = 0$ for x rational, whereas $\lim_{t \to x} f(t) = \lim_{t \to x} kt = kx \ne 0$ for x irrational. Hence, f is not continuous for all $x \ne 0$.

90. $\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

(a) $\lim_{x \to 0^-} \text{sgn}(x) = -1$

(b) $\lim_{x \to 0^+} \text{sgn}(x) = 1$

(c) $\lim_{x \to 0} \text{sgn}(x)$ does not exist.



91. True

1. $f(c) = L$ is defined.
2. $\lim_{x\to c} f(x) = L$ exists.
3. $f(c) = \lim_{x\to c} f(x)$

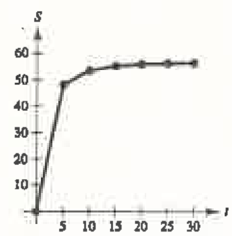
All of the conditions for continuity are met.

92. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$ and at least one of these limits (if they exist) does not equal the corresponding function at $x = c$.

93. False; a rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

94. False; $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

95. (a)



(b) There appears to be a limiting speed and a possible cause is air resistance.

96. 1. $f(x_0)$ is defined.

2. $\lim_{x\to x_0} f(x) = \lim_{\Delta x\to 0} f(x_0 + \Delta x) = f(x_0)$

(Let $x = x_0 + \Delta x$. As $x \to x_0$, $\Delta x \to 0$.)

3. $f(x_0) = \lim_{x\to x_0} f(x)$

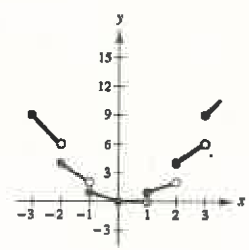
Therefore, f is continuous at x_0.

97. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

98. $h(x) = x[\![x]\!]$

h has nonremovable discontinuities at

$x = \pm 1, \pm 2, \pm 3, \ldots$



99. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}$, $c > 0$

Domain: $x + c^2 \geq 0 \implies x \geq -c^2$ and $x \neq 0$, $[-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x+c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x+c^2} - c}{x} \cdot \frac{\sqrt{x+c^2} + c}{\sqrt{x+c^2} + c}$$

$$= \lim_{x\to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x+c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x+c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

100. Define $f(x) = f_2(x) - f_1(x)$. Since f_1 and f_2 are continuous on $[a, b]$, so is f.

$$f(a) = f_2(a) - f_1(a) > 0 \quad \text{and} \quad f(b) = f_2(b) - f_1(b) < 0.$$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$$

Section 1.5 Infinite Limits

1. $\displaystyle\lim_{x\to-2^+} \frac{1}{(x+2)^2} = \infty$

$\displaystyle\lim_{x\to-2^-} \frac{1}{(x+2)^2} = \infty$

2. $\displaystyle\lim_{x\to-2^+} \frac{1}{x+2} = \infty$

$\displaystyle\lim_{x\to-2^-} \frac{1}{x+2} = -\infty$

3. $\displaystyle\lim_{x\to-2^+} \tan\frac{\pi x}{4} = -\infty$

$\displaystyle\lim_{x\to-2^-} \tan\frac{\pi x}{4} = \infty$

4. $\displaystyle\lim_{x\to-2^+} \sec\frac{\pi x}{4} = \infty$

$\displaystyle\lim_{x\to-2^-} \sec\frac{\pi x}{4} = -\infty$

5. $f(x) = \dfrac{1}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$\displaystyle\lim_{x\to-3^-} f(x) = \infty$

$\displaystyle\lim_{x\to-3^+} f(x) = -\infty$

6. $f(x) = \dfrac{x}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.077	-5.082	-50.08	-500.1	499.9	49.92	4.915	0.9091

$\displaystyle\lim_{x\to-3^-} f(x) = -\infty$

$\displaystyle\lim_{x\to-3^+} f(x) = \infty$

7. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\displaystyle\lim_{x\to-3^-} f(x) = \infty$

$\displaystyle\lim_{x\to-3^+} f(x) = -\infty$

8. $f(x) = \sec \dfrac{\pi x}{6}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	-3.864	-19.11	-191.0	-1910	1910	191.0	19.11	3.864

$\lim_{x\to -3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = \infty$

9. $\lim_{x\to 0^+} \dfrac{1}{x^2} = \infty = \lim_{x\to 0^-} \dfrac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

10. $\lim_{x\to 2^+} \dfrac{4}{(x-2)^3} = \infty$

$\lim_{x\to 2^-} \dfrac{4}{(x-2)^3} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

11. $\lim_{x\to 2^+} \dfrac{x^2-2}{(x-2)(x+1)} = \infty$

$\lim_{x\to 2^-} \dfrac{x^2-2}{(x-2)(x+1)} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

$\lim_{x\to -1^+} \dfrac{x^2-2}{(x-2)(x+1)} = \infty$

$\lim_{x\to -1^-} \dfrac{x^2-2}{(x-2)(x+1)} = -\infty$

Therefore, $x = -1$ is a vertical asymptote.

12. $\lim_{x\to 1^+} \dfrac{2+x}{1-x} = -\infty$

$\lim_{x\to 1^-} \dfrac{2+x}{1-x} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

13. $\lim_{x\to -1^+} \dfrac{x^3}{x^2-1} = \infty$

$\lim_{x\to -1^-} \dfrac{x^3}{x^2-1} = -\infty$

Therefore, $x = -1$ is a vertical asymptote.

$\lim_{x\to 1^+} \dfrac{x^3}{x^2-1} = \infty$

$\lim_{x\to 1^-} \dfrac{x^3}{x^2-1} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

14. No vertical asymptote since the denominator is never zero.

15. $f(x) = \tan 2x = \dfrac{\sin 2x}{\cos 2x}$ has vertical asymptotes at

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2},\ n \text{ any integer.}$$

16. $f(x) = \sec \pi x = \dfrac{1}{\cos \pi x}$ has vertical asymptotes at

$$x = \frac{2n+1}{2},\ n \text{ any integer.}$$

17. $\lim_{t\to 0^+} \left(1 - \dfrac{4}{t^2}\right) = -\infty = \lim_{t\to 0^-} \left(1 - \dfrac{4}{t^2}\right)$

Therefore, $t = 0$ is a vertical asymptote.

18. $\lim_{s\to 2^+} \dfrac{-2}{(s-2)^2} = -\infty = \lim_{s\to 2^-} \dfrac{-2}{(s-2)^2}$

Therefore, $s = 2$ is a vertical asymptote.

19. $\lim_{x \to -2^+} \dfrac{x}{(x+2)(x-1)} = \infty$

$\lim_{x \to -2^-} \dfrac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x \to 1^+} \dfrac{x}{(x+2)(x-1)} = \infty$

$\lim_{x \to 1^-} \dfrac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

20. $\lim_{x \to -3^+} \dfrac{1}{(x+3)^4} = \infty = \lim_{x \to -3^-} \dfrac{1}{(x+3)^4}$

Therefore, $x = -3$ is a vertical asymptote.

21. $f(x) = \dfrac{x^3+1}{x+1} = \dfrac{(x+1)(x^2-x+1)}{x+1}$

has no vertical asymptote since

$\lim_{x \to -1} f(x) = \lim_{x \to -1} (x^2 - x + 1) = 3,$

not infinity.

22. $h(x) = \dfrac{x^2-4}{x^3+2x^2+x+2} = \dfrac{(x+2)(x-2)}{(x+2)(x^2+1)}$

has no vertical asymptote since

$\lim_{x \to -2} h(x) = \lim_{x \to -2} \dfrac{x-2}{x^2+1} = -\dfrac{4}{5},$

not infinity.

23. $s(t) = \dfrac{t}{\sin t}$ has vertical asymptotes at $t = n\pi$, n a nonzero integer. There is no vertical asymptote at $t = 0$ since

$\lim_{t \to 0} \dfrac{t}{\sin t} = 1.$

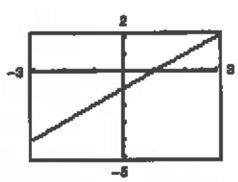
24. $g(\theta) = \dfrac{\tan\theta}{\theta} = \dfrac{\sin\theta}{\theta\cos\theta}$ has vertical asymptotes at

$\theta = \dfrac{(2n+1)\pi}{2} = \dfrac{\pi}{2} + n\pi$, n any integer.

There is no vertical asymptote at $\theta = 0$ since

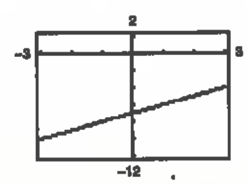
$\lim_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1.$

25. $\lim_{x \to -1} \dfrac{x^2-1}{x+1} = \lim_{x \to -1} (x-1) = -2$



Removable discontinuity at $x = -1$

26. $\lim_{x \to -1} \dfrac{x^2-6x-7}{x+1} = \lim_{x \to -1} (x-7) = -8$

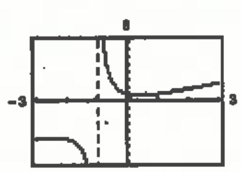


Removable discontinuity at $x = -1$

27. $\lim_{x \to -1^+} \dfrac{x^2+1}{x+1} = \infty$

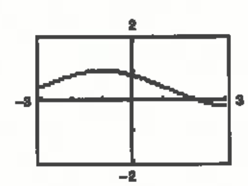
$\lim_{x \to -1^-} \dfrac{x^2+1}{x+1} = -\infty$

Vertical asymptote at $x = -1$



28. $\lim_{x \to -1} \dfrac{\sin(x+1)}{x+1} = 1$

Removable discontinuity at $x = -1$



29. $\lim_{x \to 2^+} \dfrac{x-3}{x-2} = -\infty$

30. $\lim_{x \to 1^+} \dfrac{2+x}{1-x} = -\infty$

31. $\lim_{x\to 4^+} \dfrac{x^2}{x^2-16} = \infty$

32. $\lim_{x\to 4^-} \dfrac{x^2}{x^2+16} = \dfrac{1}{2}$

33. $\lim_{x\to -3^-} \dfrac{x^2+2x-3}{x^2+x-6} = \lim_{x\to -3^-} \dfrac{x-1}{x-2} = \dfrac{4}{5}$

34. $\lim_{x\to -(1/2)^+} \dfrac{6x^2+x-1}{4x^2-4x-3} = \lim_{x\to -(1/2)^+} \dfrac{3x-1}{2x-3} = \dfrac{5}{8}$

35. $\lim_{x\to 0^-} \left(1+\dfrac{1}{x}\right) = -\infty$

36. $\lim_{x\to 0^-} \left(x^2-\dfrac{1}{x}\right) = \infty$

37. $\lim_{x\to 0^+} \dfrac{2}{\sin x} = \infty$

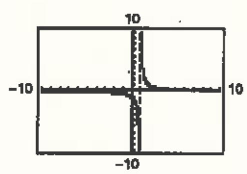
38. $\lim_{x\to (\pi/2)^+} \dfrac{-2}{\cos x} = \infty$

39. $\lim_{x\to 1} \dfrac{x^2-x}{(x^2+1)(x-1)} = \lim_{x\to 1} \dfrac{x}{x^2+1} = \dfrac{1}{2}$

40. $\lim_{x\to 3} \dfrac{x-2}{x^2} = \dfrac{1}{9}$

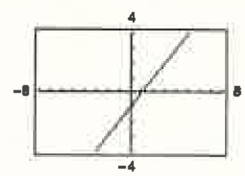
41. $f(x) = \dfrac{x^2+x+1}{x^3-1}$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$



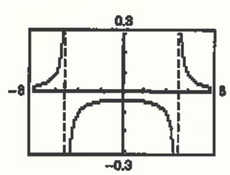
42. $f(x) = \dfrac{x^3-1}{x^2+x+1}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1) = 0$



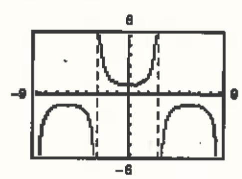
43. $f(x) = \dfrac{1}{x^2-25}$

$\lim_{x\to 5^-} f(x) = -\infty$



44. $f(x) = \sec \dfrac{\pi x}{6}$

$\lim_{x\to 3^+} f(x) = -\infty$



45. $S = \dfrac{k}{1-r}$, $0 < |r| < 1$. Assume $k \neq 0$.

$\lim_{r\to 1^-} S = \lim_{r\to 1^-} \dfrac{k}{1-r} = \infty$ (or $-\infty$ if $k < 0$)

46. $P = \dfrac{k}{V}$

$\lim_{V\to 0^+} \dfrac{k}{V} = k(\infty) = \infty$ (In this case we know that $k > 0$.)

47. (a) $r = \dfrac{2(7)}{\sqrt{625-49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625-225}} = \dfrac{3}{2}$ ft/sec

(c) $\lim_{x\to 25^-} \dfrac{2x}{\sqrt{625-x^2}} = \infty$

48. (a) $r = 50\pi \sec^2 \dfrac{\pi}{6} = \dfrac{200\pi}{3}$ ft/sec

(b) $r = 50\pi \sec^2 \dfrac{\pi}{3} = 200\pi$ ft/sec

(c) $\lim_{\theta\to (\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

49. $C = \dfrac{528x}{100 - x}, 0 \le x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\displaystyle\lim_{x \to 100^-} \frac{528}{100 - x} = \infty$ Thus, it is not possible.

50. (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

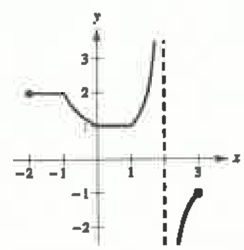
x	30	40	50	60
y	150	66.667	50	42.857

(c) $\displaystyle\lim_{x \to 25^+} \frac{25x}{x - 25} = \infty$

51. $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

52.



53. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections. The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

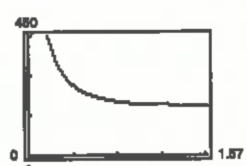
Total length $= 60 \cot \phi + 30(\pi + 2\phi)$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $\displaystyle\lim_{\phi \to (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\displaystyle\lim_{\phi \to 0^+} L = \infty$

54. (a) $A = \frac{1}{2}bh - \frac{1}{2}r^2\,\theta = \frac{1}{2}(10)(10\tan\theta) - \frac{1}{2}(10)^2\theta$

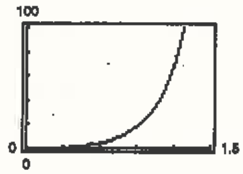
$= 50\tan\theta - 50\,\theta$

Domain: $\left(0, \frac{\pi}{2}\right)$

(b)

ϕ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1

(c)



(d) $\lim_{\theta\to\pi/2^-} A = \infty$

55. False; for instance, let

$$f(x) = \frac{x^2-1}{x-1}.$$

The graph of f has a hole at $(1, 2)$, not a vertical asymptote.

56. False; for instance, let

$$f(x) = \frac{x^2-1}{x-1} \text{ or}$$

$$g(x) = \frac{x}{x^2+1}.$$

57. True

58. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

59. Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^4}$, and $c = 0$.

$\lim_{x\to0}\frac{1}{x^2} = \infty$ and $\lim_{x\to0}\frac{1}{x^4} = \infty$, but

$$\lim_{x\to0}\left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x\to0}\left(\frac{x^2-1}{x^4}\right) = -\infty \neq 0.$$

60. Given $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$:

(2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x\to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x\to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

Therefore, $\lim_{x\to c}\frac{g(x)}{f(x)} = 0$.

61. Given $\lim_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then

$$\lim_{x\to c}\frac{g(x)}{f(x)} = \lim_{x\to c}\frac{1}{f(x)} = 0$$

by Theorem 1.15.

62. Given $\lim_{x\to c}\frac{1}{f(x)} = 0$.

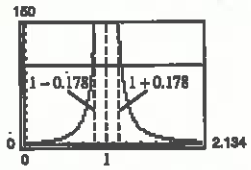
Suppose $\lim_{x\to c} f(x)$ exists and equals L. Then,

$$\lim_{x\to c}\frac{1}{f(x)} = \frac{\lim_{x\to c} 1}{\lim_{x\to c} f(x)} = \frac{1}{L} = 0.$$

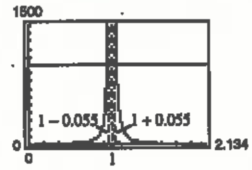
This is not possible. Thus, $\lim_{x\to c} f(x)$ does not exist.

63. $f(x) = \dfrac{x+2}{(x-1)^2}$

(a) $\delta \approx 0.178$ for $M = 100$.



(b) $\delta \approx 0.055$ for $M = 1000$.



(c) As M increases, δ decreases.

Review Exercises for Chapter 1

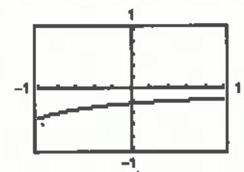
1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, 8.25.

2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.26	-0.25	-0.250	-0.2499	-0.249	-0.24

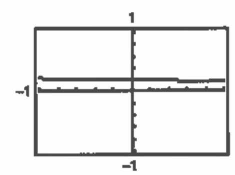
$\lim\limits_{x\to 0} f(x) \approx -0.25$



4.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.202	0.2	0.2	0.2	0.2	0.198

$\lim\limits_{x\to 0} f(x) \approx 0.2$



5. $h(x) = \dfrac{x^2 - 2x}{x}$

(a) $\lim\limits_{x\to 0} h(x) = -2$

(b) $\lim\limits_{x\to -1} h(x) = -3$

6. $g(x) = \dfrac{3x}{x-2}$

(a) $\lim\limits_{x\to 2^+} g(x) = \infty$

(b) $\lim\limits_{x\to 0} g(x) = 0$

7. $\lim\limits_{x\to 2} (5x - 3) = 5(2) - 3 = 7$

8. $\lim\limits_{x\to 2} (3x + 5) = 3(2) + 5 = 11$

9. $\lim\limits_{x\to 2} (5x-3)(3x+5) = [5(2) - 3][3(2) + 5]$
$= 7 \cdot 11 = 77$

10. $\lim\limits_{x\to 2} \left(\dfrac{3x+5}{5x-3}\right) = \dfrac{3(2)+5}{5(2)-3} = \dfrac{11}{7}$

11. $\lim\limits_{t\to 3} \dfrac{t^2+1}{t} = \dfrac{3^2+1}{3} = \dfrac{10}{3}$

12. $\lim\limits_{t\to 3} \dfrac{t^2-9}{t-3} = \lim\limits_{t\to 3} (t+3) = 6$

13. $\lim\limits_{t\to -2} \dfrac{t+2}{t^2-4} = \lim\limits_{t\to -2} \dfrac{1}{t-2} = -\dfrac{1}{4}$

14. $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x} - 2}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt{4+x} - 2}{x} \cdot \dfrac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$
$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{4+x} + 2} = \dfrac{1}{4}$

15. $\lim\limits_{x\to 0} \dfrac{[1/(x+1)] - 1}{x} = \lim\limits_{x\to 0} \dfrac{1 - (x+1)}{x(x+1)}$
$= \lim\limits_{x\to 0} \dfrac{-1}{x+1} = -1$

16. $\lim_{s\to0}\frac{\left(1/\sqrt{1+s}\right)-1}{s}=\lim_{s\to0}\left[\frac{\left(1/\sqrt{1+s}\right)-1}{s}\cdot\frac{\left(1/\sqrt{1+s}\right)+1}{\left(1/\sqrt{1+s}\right)+1}\right]$

$=\lim_{s\to0}\frac{[1/(1+s)]-1}{s\left[\left(1/\sqrt{1+s}\right)+1\right]}=\lim_{s\to0}\frac{-1}{(1+s)\left[\left(1/\sqrt{1+s}\right)+1\right]}=-\frac{1}{2}$

17. $\lim_{x\to-5}\frac{x^3+125}{x+5}=\lim_{x\to-5}\frac{(x+5)(x^2-5x+25)}{x+5}$

$=\lim_{x\to-5}(x^2-5x+25)$

$=75$

18. $\lim_{x\to-2}\frac{x^2-4}{x^3+8}=\lim_{x\to-2}\frac{(x+2)(x-2)}{(x+2)(x^2-2x+4)}$

$=\lim_{x\to-2}\frac{x-2}{x^2-2x+4}$

$=-\frac{4}{12}=-\frac{1}{3}$

19. $\lim_{x\to0^+}\left(x-\frac{1}{x^3}\right)=-\infty$

20. $\lim_{x\to2^+}\frac{1}{\sqrt[3]{x^2-4}}=\infty$

$\lim_{x\to2^-}\frac{1}{\sqrt[3]{x^2-4}}=-\infty$

Thus, $\lim_{x\to2}\frac{1}{\sqrt[3]{x^2-4}}$ does not exist.

21. $\lim_{\Delta x\to0}\frac{\sin[(\pi/6)+\Delta x]-(1/2)}{\Delta x}=\lim_{\Delta x\to0}\frac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}$

$=\lim_{\Delta x\to0}\frac{1}{2}\cdot\frac{(\cos\Delta x-1)}{\Delta x}+\lim_{\Delta x\to0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x}$

$=0+\frac{\sqrt{3}}{2}(1)=\frac{\sqrt{3}}{2}$

22. $\lim_{\Delta x\to0}\frac{\cos(\pi+\Delta x)+1}{\Delta x}=\lim_{\Delta x\to0}\frac{\cos\pi\cos\Delta x-\sin\pi\sin\Delta x+1}{\Delta x}$

$=\lim_{\Delta x\to0}\left[-\frac{(\cos\Delta x-1)}{\Delta x}\right]-\lim_{\Delta x\to0}\left[\sin\pi\frac{\sin\Delta x}{\Delta x}\right]$

$=-0-(0)(1)=0$

23. $\lim_{x\to-2^-}\frac{2x^2+x+1}{x+2}=-\infty$

24. $\lim_{x\to(1/2)^+}\frac{x}{2x-1}=\infty$

25. $\lim_{x\to-1^+}\frac{x+1}{x^3+1}=\lim_{x\to-1^+}\frac{1}{x^2-x+1}=\frac{1}{3}$

26. $\lim_{x\to-1^-}\frac{x+1}{x^4-1}=\lim_{x\to-1^-}\frac{1}{(x^2+1)(x-1)}=-\frac{1}{4}$

27. $\lim_{x\to1^-}\frac{x^2+2x+1}{x-1}=-\infty$

28. $\lim_{x\to-1^+}\frac{x^2-2x+1}{x+1}=\infty$

29. $\lim_{x\to0^+}\frac{\sin 4x}{5x}=\lim_{x\to0^+}\left[\frac{4}{5}\left(\frac{\sin 4x}{4x}\right)\right]=\frac{4}{5}$

30. $\lim_{x\to0^+}\frac{\sec x}{x}=\infty$

31. $\lim_{x\to0^+}\frac{\csc 2x}{x}=\lim_{x\to0^+}\frac{1}{x\sin 2x}=\infty$

32. $\lim_{x\to0^-}\frac{\cos^2 x}{x}=-\infty$

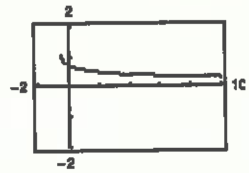
33. $f(x) = \dfrac{\sqrt{2x+1} - \sqrt{3}}{x-1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$$\lim_{x\to 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \approx 0.577 \quad \left(\text{Actual limit is } \sqrt{3}/3.\right)$$

(b)



(c) $$\lim_{x\to 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} = \lim_{x\to 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{2x+1} + \sqrt{3}}{\sqrt{2x+1} + \sqrt{3}}$$

$$= \lim_{x\to 1^+} \frac{(2x+1) - 3}{(x-1)\left(\sqrt{2x+1} + \sqrt{3}\right)}$$

$$= \lim_{x\to 1^+} \frac{2}{\sqrt{2x+1} + \sqrt{3}}$$

$$= \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

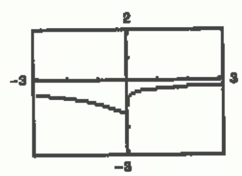
34. $f(x) = \dfrac{1 - \sqrt[3]{x}}{x-1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	−0.3228	−0.3322	−0.3332	−0.3333

$$\lim_{x\to 1^+} \frac{1 - \sqrt[3]{x}}{x-1} \approx -0.333 \quad \left(\text{Actual limit is } -\tfrac{1}{3}.\right)$$

(b)



(c) $$\lim_{x\to 1^+} \frac{1 - \sqrt[3]{x}}{x-1} = \lim_{x\to 1^+} \frac{1 - \sqrt[3]{x}}{x-1} \cdot \frac{1 + \sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2}{1 + \sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2}$$

$$= \lim_{x\to 1^+} \frac{1-x}{(x-1)\left[1 + \sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right]}$$

$$= \lim_{x\to 1^+} \frac{-1}{1 + \sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2}$$

$$= -\frac{1}{3}$$

35. $f(x) = [\![x + 3]\!]$

$\lim\limits_{x\to k^+} [\![x + 3]\!] = k + 3$ where k is an integer.

$\lim\limits_{x\to k^-} [\![x + 3]\!] = k + 2$ where k is an integer.

Nonremovable discontinuity at each integer k

Continuous on $(k, k + 1)$ for all integers k

36. $f(x) = \dfrac{3x^2 - x - 2}{x-1} = \dfrac{(3x+2)(x-1)}{x-1}$

$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} (3x + 2) = 5$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

37. $f(x) = \begin{cases} \dfrac{3x^2 - x - 2}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{3x^2 - x - 2}{x-1}$$

$$= \lim_{x\to 1} (3x + 2) = 5 \neq 0$$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

38. $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim\limits_{x\to 2^-} (5 - x) = 3$

$\lim\limits_{x\to 2^+} (2x - 3) = 1$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

39. $f(x) = \dfrac{1}{(x-2)^2}$

$\lim_{x \to 2} \dfrac{1}{(x-2)^2} = \infty$

Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

40. $f(x) = \sqrt{\dfrac{x+1}{x}} = \sqrt{1 + \dfrac{1}{x}}$

$\lim_{x \to 0^+} \sqrt{1 + \dfrac{1}{x}} = \infty$

Domain: $(-\infty, -1], (0, \infty)$

Nonremovable discontinuity at $x = 0$
Continuous on $(-\infty, -1] \cup (0, \infty)$

41. $f(x) = \dfrac{3}{x+1}$

$\lim_{x \to 1^-} f(x) = -\infty$

$\lim_{x \to 1^+} f(x) = \infty$

Nonremovable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

42. $f(x) = \dfrac{x+1}{2x+2}$

$\lim_{x \to -1} \dfrac{x+1}{2(x+1)} = \dfrac{1}{2}$

Removable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

43. $f(x) = \csc \dfrac{\pi x}{2}$

Nonremovable discontinuities at each even integer. Continuous on

$(2k, 2k + 2)$

for all integers k.

44. $f(x) = \tan 2x$

Nonremovable discontinuities when

$x = \dfrac{(2n+1)\pi}{4}$

Continuous on

$\left(\dfrac{(2n-1)\pi}{4}, \dfrac{(2n+1)\pi}{4}\right)$

for all integers n.

45. $f(2) = 5$

Find c so that $\lim_{x \to 2^+} (cx + 6) = 5$.

$c(2) + 6 = 5$

$2c = -1$

$c = -\dfrac{1}{2}$

46. $\lim_{x \to 1^+} (x + 1) = 2$

$\lim_{x \to 3^-} (x + 1) = 4$

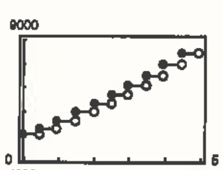
Find b and c so that $\lim_{x \to 1^-} (x^2 + bx + c) = 2$ and $\lim_{x \to 3^+} (x^2 + bx + c) = 4$.

Consequently we get $1 + b + c = 2$ and $9 + 3b + c = 4$.

Solving simultaneously, $b = -3$ and $c = 4$.

47. $A = 5000(1.06)^{[\![2t]\!]}$

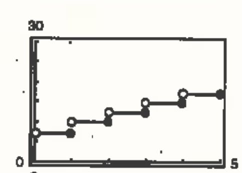
Nonremovable discontinuity every 6 months



48. $C = 9.80 + 2.50[-[\![-x]\!] - 1], \ x > 0$

$= 9.80 - 2.50[[\![-x]\!] + 1]$

C has a nonremovable discontinuity at each integer.



49. $g(x) = 1 + \dfrac{2}{x}$

Vertical asymptote at $x = 0$

50. $h(x) = \dfrac{4x}{4 - x^2}$

Vertical asymptotes at $x = 2$ and $x = -2$

51. $f(x) = \dfrac{8}{(x - 10)^2}$

Vertical asymptote at $x = 10$

52. $f(x) = \csc \pi x$

Vertical asymptote at every integer k

53. $C = \dfrac{80{,}000p}{100 - p}$, $0 \le p < 100$

(a) $C(15) \approx \$14{,}117.65$ (b) $C(50) = \$80{,}000$

(c) $C(90) = \$720{,}000$ (d) $\displaystyle\lim_{p \to 100^-} \frac{80{,}000p}{100 - p} = \infty$

54. $f(x) = \dfrac{\tan 2x}{x}$

(a)

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$$\lim_{x \to 0} \frac{\tan 2x}{x} = 2$$

(b) Yes, define

$$f(x) = \begin{cases} \dfrac{\tan 2x}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}.$$

Now $f(x)$ is continuous at $x = 0$.

55. $$\lim_{t \to a} \frac{s(a) - s(t)}{a - t} = \lim_{t \to 4} \frac{(-4.9(4)^2 + 200) - (-4.9t^2 + 200)}{4 - t}$$
$$= \lim_{t \to 4} \frac{4.9(t - 4)(t + 4)}{4 - t}$$
$$= \lim_{t \to 4} -4.9(t + 4) = -39.2 \text{ m/sec}$$

56. $s(t) = 0 \implies -4.9t^2 + 200 = 0 \implies t^2 \approx 40.816 \implies t \approx 6.39$ sec

When $t = 6.39$, the velocity is approximately

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t} = \lim_{t \to a} -4.9(a + 4)$$
$$= \lim_{t \to 6.39} -4.9(6.39 + 4) = -50.9 \text{ m/sec.}$$

57. $\displaystyle\lim_{x \to 0} \frac{|x|}{x} = 1$ is false. $\displaystyle\lim_{x \to 0} \frac{|x|}{x}$ does not exist, since

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

58. $\displaystyle\lim_{x \to 0} x^3 = 0$ is true.

59. True; see Theorem 1.7.

60. False; let $f(x) = \dfrac{x - 1}{x^2 - 1}$. Then,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1}{x + 1} = \frac{1}{2}$$

but $f(1)$ is not defined.

61. True

62. $\lim_{x\to 2} f(x) = 3$ is false since

$\lim_{x\to 2^-} f(x) = 3$ and $\lim_{x\to 2^+} f(x) = 0$.

63. $\lim_{x\to 3} f(x) = 1$ is true since

$\lim_{x\to 3^-} (x - 2) = 1$ and $\lim_{x\to 3^+} (-x^2 + 8x - 14) = 1$.

64. $f(x) = \dfrac{x^2 - 4}{|x - 2|} = (x + 2)\left[\dfrac{x - 2}{|x - 2|}\right]$

$\lim_{x\to 2^-} f(x) = -4$

$\lim_{x\to 2^+} f(x) = 4$

$\lim_{x\to 2} f(x)$ does not exist.

65. $f(x) = \sqrt{(x - 1)x}$

Domain: $(-\infty, 0] \cup [1, \infty)$

$\lim_{x\to 0^-} f(x) = 0$

$\lim_{x\to 1^+} f(x) = 0$

CHAPTER 2
Differentiation

CHAPTER 2
Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Exercises

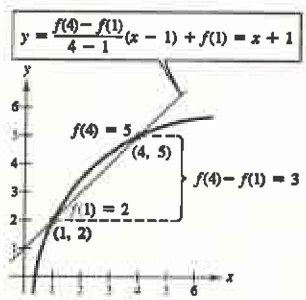
1. (a) $m = 0$

(b) $m = -3$

2. (a) $m = \frac{1}{4}$

(b) $m = 1$

3. (a), (b)



(c) $$y = \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$$

$$= \frac{3}{3}(x - 1) + 2$$

$$= 1(x - 1) + 2$$

$$= x + 1$$

4. (a) $$\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$$

$$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$$

Thus, $\frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

5. $f(x) = 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3 - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0 = 0$$

6. $f(x) = 3x + 2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3 = 3$$

7. $f(x) = -5x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -5 = -5$$

8. $f(x) = 9 - \frac{1}{2}x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

9. $f(x) = 2x^2 + x - 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2\Delta x + 1) = 4x + 1$$

10. $f(x) = 1 - x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (-2x - \Delta x) = -2x$$

11. $f(x) = x^3 - 12x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^3 - 12$$

12. $f(x) = x^3 + x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x$$

13. $f(x) = \dfrac{1}{x-1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x+\Delta x-1} - \dfrac{1}{x-1}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x-1) - (x+\Delta x - 1)}{\Delta x(x+\Delta x-1)(x-1)}$$
$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x+\Delta x-1)(x-1)}$$
$$= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x-1)(x-1)}$$
$$= -\frac{1}{(x-1)^2}$$

14. $f(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x+\Delta x)^2} - \dfrac{1}{x^2}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 - (x+\Delta x)^2}{\Delta x(x+\Delta x)^2 x^2}$$
$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x+\Delta x)^2 x^2}$$
$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x+\Delta x)^2 x^2}$$
$$= \frac{-2x}{x^4}$$
$$= -\frac{2}{x^3}$$

15. $f(x) = \sqrt{x-4}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x-4} - \sqrt{x-4}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x-4} + \sqrt{x-4}}{\sqrt{x+\Delta x-4} + \sqrt{x-4}}$$
$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x-4) - (x-4)}{\Delta x\left[\sqrt{x+\Delta x-4} + \sqrt{x-4}\right]} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x-4} + \sqrt{x-4}} = \frac{1}{2\sqrt{x-4}}$$

16. $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{\sqrt{x+\Delta x}} - \dfrac{1}{\sqrt{x}}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\sqrt{x} - \sqrt{x+\Delta x}}{\Delta x\sqrt{x+\Delta x}\sqrt{x}} \cdot \frac{\sqrt{x} + \sqrt{x+\Delta x}}{\sqrt{x} + \sqrt{x+\Delta x}} = \lim_{\Delta x \to 0} \frac{x - (x+\Delta x)}{\Delta x\sqrt{x+\Delta x}\sqrt{x}\left[\sqrt{x} + \sqrt{x+\Delta x}\right]}$$
$$= \lim_{\Delta x \to 0} \frac{-1}{\sqrt{x+\Delta x}\sqrt{x}\left[\sqrt{x} + \sqrt{x+\Delta x}\right]} = -\frac{1}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = -\frac{1}{2x\sqrt{x}}$$

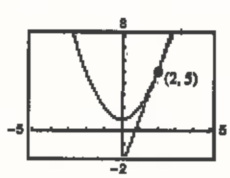
17. (a) $f(x) = x^2 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 + 1] - [x^2 + 1]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $(2, 5)$, the slope of the tangent line is $m = 2(2) = 4$. The equation of the tangent line is

$$y - 5 = 4(x - 2)$$
$$y - 5 = 4x - 8$$
$$y = 4x - 3.$$

(b)



18. (a) $f(x) = x^2 + 2x + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 2$$

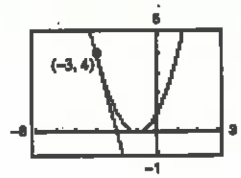
At $(-3, 4)$, the slope of the tangent line is

$$m = 2(-3) + 2 = -4.$$

The equation of the tangent line is

$$y - 4 = -4(x + 3)$$
$$y = -4x - 8.$$

(b)



19. (a) $f(x) = x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

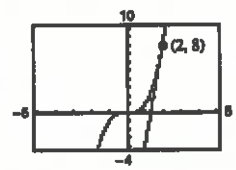
$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$.
The equation of the tangent line is

$$y - 8 = 12(x - 2)$$
$$y = 12x - 16.$$

(b)



20. (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

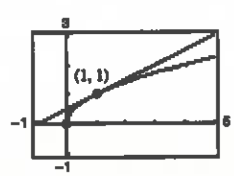
At $(1, 1)$, the slope of the tangent line is

$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$
$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)



21. (a) $f(x) = x + \frac{1}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x) + \frac{1}{x + \Delta x}\right] - \left[x + \frac{1}{x}\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x + \frac{x - (x + \Delta x)}{(x + \Delta x)x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[1 + \frac{-\Delta x}{\Delta x(x + \Delta x)x}\right]$$

$$= \lim_{\Delta x \to 0} \left[1 - \frac{1}{(x + \Delta x)x}\right]$$

$$= 1 - \frac{1}{x^2}$$

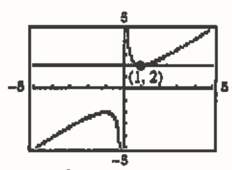
At $(1, 2)$, the slope of the tangent line is

$$m = 1 - \frac{1}{1^2} = 0.$$

The equation of the tangent line is

$$y - 2 = 0(x - 1)$$
$$y = 2.$$

(b)



22. (a) $f(x) = \frac{1}{x + 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x + 1} - \frac{1}{x + 1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + 1) - (x + \Delta x + 1)}{\Delta x(x + \Delta x + 1)(x + 1)}$$

$$= \lim_{\Delta x \to 0} -\frac{1}{(x + \Delta x + 1)(x + 1)}$$

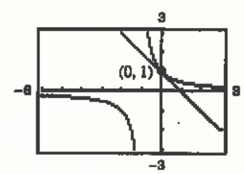
$$= -\frac{1}{(x + 1)^2}$$

At $(0, 1)$, the slope of the tangent line is

$$m = \frac{-1}{(0 + 1)^2} = -1.$$

The equation of the tangent line is $y = -x + 1$.

(b)



23. From Exercise 19 we know that $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$
$$x = \pm 1.$$

Therefore, at the points $(1, 1)$ and $(-1, -1)$ the tangent lines are parallel to $3x - y + 1 = 0$. These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$
$$y = 3x - 2 \qquad\qquad y = 3x + 2.$$

24. From Exercise 16 we know that

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is $-\frac{1}{2}$, we have

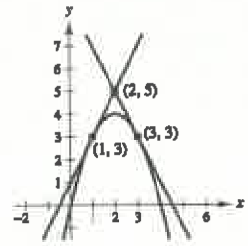
$$-\frac{1}{2x\sqrt{x}} = -\frac{1}{2}$$
$$x = 1.$$

Therefore, at the point $(1, 1)$ the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$
$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$
$$y = -\frac{1}{2}x + \frac{3}{2}.$$

25. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through (2, 5) and (x_0, y_0) equals the derivative of f at x_0:

$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$
$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$
$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$
$$0 = x_0^2 - 4x_0 + 3$$
$$0 = (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3$$

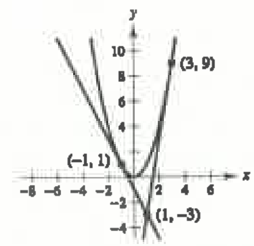


Therefore, the points of tangency are (1, 3) and (3, 3), and the corresponding slopes are 2 and -2. The equations of the tangent lines are

$$y - 5 = 2(x - 2) \qquad y - 5 = -2(x - 2)$$
$$y = 2x + 1 \qquad y = -2x + 9$$

26. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of f at x_0:

$$\frac{-3 - y_0}{1 - x_0} = 2x_0$$
$$-3 - y_0 = (1 - x_0)2x_0$$
$$-3 - x_0^2 = 2x_0 - 2x_0^2$$
$$x_0^2 - 2x_0 - 3 = 0$$
$$(x_0 - 3)(x_0 + 1) = 0 \Rightarrow x_0 = 3, -1$$



Therefore, the points of tangency are (3, 9) and $(-1, 1)$, and the corresponding slopes are 6 and -2. The equations of the tangent lines are

$$y + 3 = 6(x - 1) \qquad y + 3 = -2(x - 1)$$
$$y = 6x - 9 \qquad y = -2x - 1$$

27. $f(x) = x$

$f'(x) = 1$

Matches graph (b).

28. $f(x) = x^2$

$f'(x) = 2x$

Matches graph (e).

29. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Matches graph (c).

30. $f(x) = \dfrac{1}{x}$

$f'(x) = -\dfrac{1}{x^2}$

Matches graph (a).

31. $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$f'(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$

Matches graph (f).

32. $f(x) = \sin x$

The slope, $f'(x)$, is periodic. Matches graph (d).

33. (a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.

(d) Because $g'(-4) = \frac{7}{3}$, g is increasing (rising) at $x = -4$.

(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

34. (a) $f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

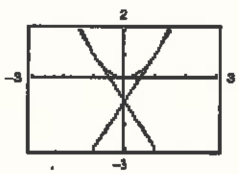
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $x = -1, f'(-1) = -2$ and the tangent line is

$y - 1 = -2(x + 1)$ or $y = -2x - 1$.

At $x = 0, f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1, f'(1) = 2$ and the tangent line is $y = 2x - 1$.



For this function, the slopes of the tangent lines are always distinct for different values of x.

(b) $g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2$$

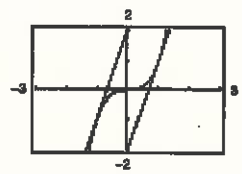
At $x = -1, g'(-1) = 3$ and the tangent line is

$y + 1 = 3(x + 1)$ or $y = 3x + 2$.

At $x = 0, g'(0) = 0$ and the tangent line is $y = 0$.

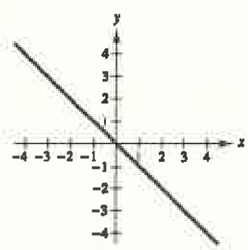
At $x = 1, g'(1) = 3$ and the tangent line is

$y - 1 = 3(x - 1)$ or $y = 3x - 2$.



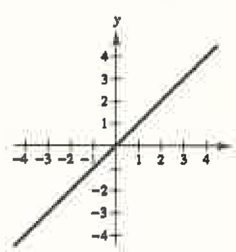
For this function, the slopes of the tangent lines are sometimes the same.

35.



$y = -x$ is one possible answer. (slope is -1)

36.

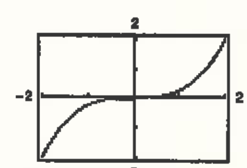


$y = x$ is one possible answer. (slope is 1)

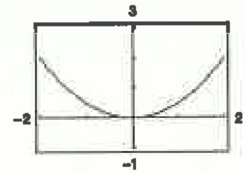
37. $f(x) = \frac{1}{4}x^3$

By the limit definition of the derivative we have $f'(x) = \frac{3}{4}x^2$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3



38. $f(x) = \frac{1}{2}x^2$

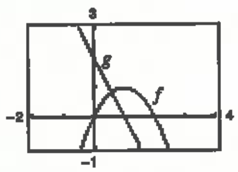


By the limit definition of the derivative we have $f'(x) = x$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	2	1.125	0.5	0.125	0	0.125	0.5	1.125	2
$f'(x)$	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2

39. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

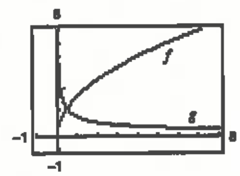
$= (2(x + 0.01) - (x + 0.01)^2 - 2x + x^2) + 100$



The graph of $g(x)$ is approximately the graph of $f'(x)$.

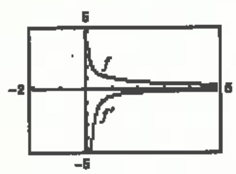
40. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= \left(3\sqrt{x + 0.01} - 3\sqrt{3}\right)100$

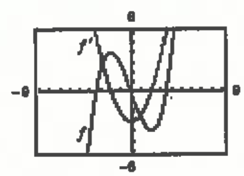


The graph of $g(x)$ is approximately the graph of $f'(x)$.

41. $f(x) = \dfrac{1}{\sqrt{x}}$ and $f'(x) = \dfrac{-1}{2x^{3/2}}$.



42. $f(x) = \dfrac{x^3}{4} - 3x$ and $f'(x) = \dfrac{3}{4}x^2 - 3$



43. $f(x) = 4 - (x - 3)^2$

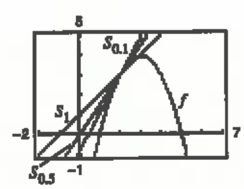
$S_{\Delta x}(x) = \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2)$

$= \dfrac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x}(x - 2) + 3 = \dfrac{1 - (\Delta x - 1)^2}{\Delta x}(x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3$

(a) $\Delta x = 1$: $S_{\Delta x} = (x - 2) + 3 = x + 1$

$\Delta x = 0.5$: $S_{\Delta x} = \left(\frac{3}{2}\right)(x - 2) + 3 = \frac{3}{2}x$

$\Delta x = 0.1$: $S_{\Delta x} = \left(\frac{19}{10}\right)(x - 2) + 3 = \frac{19}{10}x - \frac{4}{5}$



(b) As $\Delta x \to 0$, the line approaches the tangent line to f at $(2, 3)$.

44. $f(x) = x + \dfrac{1}{x}$

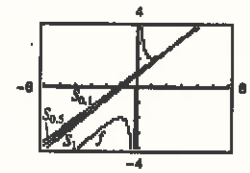
$$S_{\Delta x}(x) = \frac{f(2+\Delta x) - f(2)}{\Delta x}(x-2) + f(2) = \frac{(2+\Delta x) + \dfrac{1}{2+\Delta x} - \dfrac{5}{2}}{\Delta x}(x-2) + \frac{5}{2}$$

$$= \frac{2(2+\Delta x)^2 + 2 - 5(2+\Delta x)}{2(2+\Delta x)\,\Delta x}(x-2) + \frac{5}{2} = \frac{(2\Delta x + 3)}{2(2+\Delta x)}(x-2) + \frac{5}{2}$$

(a) $\Delta x = 1$: $S_{\Delta x} = \dfrac{5}{6}(x-2) + \dfrac{5}{2} = \dfrac{5}{6}x + \dfrac{5}{6}$

$\Delta x = 0.5$: $S_{\Delta x} = \dfrac{4}{5}(x-2) + \dfrac{5}{2} = \dfrac{4}{5}x + \dfrac{9}{10}$

$\Delta x = 0.1$: $S_{\Delta x} = \dfrac{16}{21}(x-2) + \dfrac{5}{2} = \dfrac{16}{21}x + \dfrac{41}{42}$



(b) As $\Delta x \to 0$, the line approaches the tangent line to f at $\left(2, \frac{5}{2}\right)$.

45. $f(x) = x^2 - 1, c = 2$

$$f'(2) = \lim_{x\to 2}\frac{f(x) - f(2)}{x-2} = \lim_{x\to 2}\frac{(x^2-1)-3}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

46. $f(x) = x^3 + 2x, c = 1$

$$f'(1) = \lim_{x\to 1}\frac{f(x) - f(1)}{x-1} = \lim_{x\to 1}\frac{x^3+2x-3}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+3)}{x-1} = \lim_{x\to 1}(x^2+x+3) = 5$$

47. $f(x) = x^3 + 2x^2 + 1, c = -2$

$$f'(-2) = \lim_{x\to -2}\frac{f(x) - f(-2)}{x+2} = \lim_{x\to -2}\frac{x^2(x+2)}{x+2} = \lim_{x\to -2}\frac{(x^3+2x^2+1)-1}{x+2} = \lim_{x\to -2}x^2 = 4$$

48. $f(x) = \dfrac{1}{x}, c = 3$

$$f'(3) = \lim_{x\to 3}\frac{f(x) - f(3)}{x-3} = \lim_{x\to 3}\frac{(1/x)-(1/3)}{x-3} = \lim_{x\to 3}\frac{3-x}{3x}\cdot\frac{1}{x-3} = \lim_{x\to 3}\left(-\frac{1}{3x}\right) = -\frac{1}{9}$$

49. $f(x) = (x-1)^{2/3}, c = 1$

$$f'(1) = \lim_{x\to 1}\frac{f(x) - f(1)}{x-1} = \lim_{x\to 1}\frac{(x-1)^{2/3} - 0}{x-1} = \lim_{x\to 1}\frac{1}{(x-1)^{1/3}}$$

The limit does not exist. Thus, f is not differentiable at $x = 1$.

50. $f(x) = |x-2|, c = 2$

$$f'(2) = \lim_{x\to 2}\frac{f(x) - f(2)}{x-2} = \lim_{x\to 2}\frac{|x-2|}{x-2}$$

The limit does not exist. Thus, f is not differentiable at $x = 2$.

51. $f(x)$ is differentiable everywhere except at $x = -3$. (Sharp turn in the graph.)

52. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph.)

53. $f(x)$ is differentiable everywhere except at $x = -1$. (Discontinuity)

54. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

55. $f(x)$ is differentiable everywhere except at $x = 3$. (Sharp turn in the graph)

56. $f(x)$ is differentiable everywhere except at $x = 0$. (Sharp turn in the graph)

57. $f(x)$ is differentiable on the interval $(1, \infty)$. (At $x = 1$ the tangent line is vertical.)

58. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

59. $f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

60. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

61. $f(x) = |x - 1|$

The derivative from the left is $\lim\limits_{x\to 1^-} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1^-} \dfrac{|x - 1| - 0}{x - 1} = -1.$

The derivative from the right is $\lim\limits_{x\to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1^+} \dfrac{|x - 1| - 0}{x - 1} = 1.$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

62. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x\to 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x\to 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty. \text{ (Vertical tangent)}$$

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $x = 1$.

63. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{(x - 1)^3 - 0}{x - 1} = \lim_{x\to 1^-} (x - 1)^2 = 0.$$

The derivative from the right is

$$\lim_{x\to 1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{(x - 1)^2 - 0}{x - 1} = \lim_{x\to 1^+} (x - 1) = 0.$$

These one-sided limits are equal. Therefore, f is differentiable at $x = 1$. $(f'(1) = 0)$

64. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x - 1}{x - 1} = \lim_{x\to 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x\to 1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

65. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x\to 2^-} (x + 2) = 4.$$

The derivative from the right is

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x\to 2^+} 4 = 4.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $(f'(2) = 4)$

66. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

The derivative from the left is

$$\lim_{x\to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^-} \frac{\left(\frac{1}{2}x + 1\right) - 2}{x - 2} = \lim_{x\to 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

The derivative from the right is

$$\lim_{x\to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2}$$

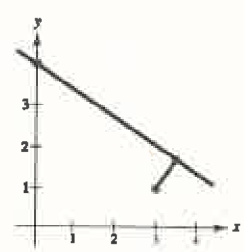
$$= \lim_{x\to 2^+} \frac{2x - 4}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2(x - 2)}{(x - 2)\left(\sqrt{2x} + 2\right)} = \lim_{x\to 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $\left(f'(2) = \frac{1}{2}\right)$

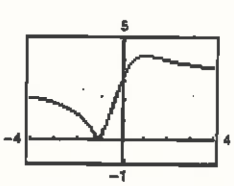
67. (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

$$= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}.$$



(b)



The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.

68. (a) If f is odd and $f'(c) = 3$, then by symmetry, $f'(-c) = 3$ also.

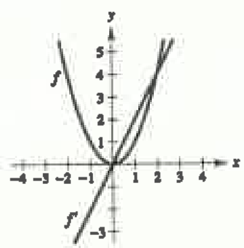
(b) If f is even and $f'(c) = 3$, then by symmetry, $f'(-c) = -3$.

69. False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

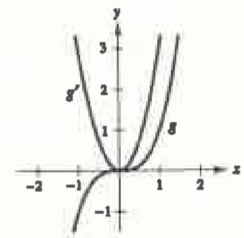
70. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is -1 and the derivative from the right at $x = 0$ is 1. At $x = 0$, the derivative does not exist.

71. True—see Theorem 2.1

72. (a) $f(x) = x^2$ and $f'(x) = 2x$



(b) $g(x) = x^3$ and $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then
$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{(x+\Delta x)^4 - x^4}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x}$$
$$= \lim_{\Delta x\to 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x}$$
$$= \lim_{\Delta x\to 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3$$

Hence, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof, since you must verify the conjecture for all integer values of n, $n \ge 2$.

73. $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, we have $-|x| \le x\sin(1/x) \le |x|$, $x \ne 0$. Thus, $\lim_{x\to 0} x\sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative we have

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0}\left(\sin\frac{1}{x}\right).$$

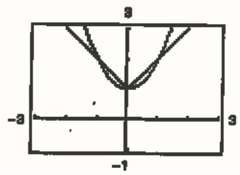
Since this limit does not exist (it oscillates between -1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have $-x^2 \le x^2\sin(1/x) \le x^2$, $x \ne 0$. Thus, $\lim_{x\to 0} x^2\sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative again we have

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

74. 

As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.

Section 2.2 Basic Differentiation Rules and Rates of Change

1. (a) $y = x^{1/2}$, $y' = \frac{1}{2}x^{-1/2}$, $y'(1) = \frac{1}{2}$
   (b) $y = x^{3/2}$, $y' = \frac{3}{2}x^{1/2}$, $y'(1) = \frac{3}{2}$
   (c) $y = x^2$, $y' = 2x$, $y'(1) = 2$
   (d) $y = x^3$, $y' = 3x^2$, $y'(1) = 3$

2. (a) $y = x^{-1/2}$, $y' = -\frac{1}{2}x^{-3/2}$, $y'(1) = -\frac{1}{2}$
   (b) $y = x^{-1}$, $y' = -x^{-2}$, $y'(1) = -1$
   (c) $y = x^{-3/2}$, $y' = -\frac{3}{2}x^{-5/2}$, $y'(1) = -\frac{3}{2}$
   (d) $y = x^{-2}$, $y' = -2x^{-3}$, $y'(1) = -2$

3. $y = 3$
   $y' = 0$

4. $f(x) = -2$
   $f'(x) = 0$

5. $f(x) = x + 1$
   $f'(x) = 1$

6. $g(x) = 3x - 1$
   $g'(x) = 3$

7. $g(x) = x^2 + 4$
   $g'(x) = 2x$

8. $y = t^2 + 2t - 3$
   $y' = 2t + 2$

9. $f(t) = -2t^2 + 3t - 6$
   $f'(t) = -4t + 3$

10. $y = x^3 - 9$
    $y' = 3x^2$

11. $s(t) = t^3 - 2t + 4$
    $s'(t) = 3t^2 - 2$

12. $f(x) = 2x^3 - x^2 + 3x$
    $f'(x) = 6x^2 - 2x + 3$

13. $y = x^2 - \frac{1}{2}\cos x$
    $y' = 2x + \frac{1}{2}\sin x$

14. $y = 5 + \sin x$
    $y' = \cos x$

15. $y = \frac{1}{x} - 3\sin x$
    $y' = -\frac{1}{x^2} - 3\cos x$

16. $g(t) = \pi\cos t$
    $g'(t) = -\pi\sin t$

Function	*Rewrite*	*Derivative*	*Simplify*
17. $y = \frac{1}{3x^3}$	$y = \frac{1}{3}x^{-3}$	$y' = -x^{-4}$	$y' = -\frac{1}{x^4}$
18. $y = \frac{2}{3x^2}$	$y = \frac{2}{3}x^{-2}$	$y' = -\frac{4}{3}x^{-3}$	$y' = -\frac{4}{3x^3}$
19. $y = \frac{1}{(3x)^3}$	$y = \frac{1}{27}x^{-3}$	$y' = -\frac{1}{9}x^{-4}$	$y' = -\frac{1}{9x^4}$
20. $y = \frac{\pi}{(3x)^2}$	$y = \frac{\pi}{9}x^{-2}$	$y' = -\frac{2\pi}{9}x^{-3}$	$y' = -\frac{2\pi}{9x^3}$

—CONTINUED—

—CONTINUED—

Function	Rewrite	Derivative	Simplify
21. $y = \dfrac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\frac{1}{2}x^{-3/2}$	$y' = -\dfrac{1}{2x^{3/2}}$
22. $y = \dfrac{4}{x^{-3}}$	$y = 4x^3$	$y' = 12x^2$	$y' = 12x^2$

23. $f(x) = \dfrac{1}{x}$, $(1, 1)$

$f'(x) = -\dfrac{1}{x^2}$

$f'(1) = -1$

24. $f(t) = 3 - \dfrac{3}{5t}$, $\left(\dfrac{3}{5}, 2\right)$

$f'(t) = \dfrac{3}{5t^2}$

$f'\left(\dfrac{3}{5}\right) = \dfrac{5}{3}$

25. $f(x) = -\dfrac{1}{2} + \dfrac{7}{5}x^3$, $\left(0, -\dfrac{1}{2}\right)$

$f'(x) = \dfrac{21}{5}x^2$

$f'(0) = 0$

26. $y = 3x\left(x^2 - \dfrac{2}{x}\right)$, $(2, 18)$

$= 3x^3 - 6$

$y' = 9x^2$

$y'(2) = 36$

27. $y = (2x + 1)^2$, $(0, 1)$

$= 4x^2 + 4x + 1$

$y' = 8x + 4$

$y'(0) = 4$

28. $f(x) = 3(5 - x)^2$, $(5, 0)$

$= 3x^2 - 30x + 75$

$f'(x) = 6x - 30$

$f'(5) = 0$

29. $f(\theta) = 4 \sin \theta - \theta$, $(0, 0)$

$f'(\theta) = 4 \cos \theta - 1$

$f'(0) = 4(1) - 1 = 3$

30. $g(t) = 2 + 3 \cos t$, $(\pi, -1)$

$g'(t) = -3 \sin t$

$g'(\pi) = 0$

31. $f(x) = x^3 - 3x - 2x^{-4}$

$f'(x) = 3x^2 - 3 + 8x^{-5}$

$= 3x^2 - 3 + \dfrac{8}{x^5}$

32. $f(x) = x^2 - 3x - 3x^{-2}$

$f'(x) = 2x - 3 + 6x^{-3}$

$= 2x - 3 + \dfrac{6}{x^3}$

33. $g(t) = t^2 - 4t^{-1}$

$g'(t) = 2t + 4t^{-2}$

$= 2t + \dfrac{4}{t^2}$

34. $f(x) = x + x^{-2}$

$f'(x) = 1 - 2x^{-3}$

$= 1 - \dfrac{2}{x^3}$

35. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3 - 8}{x^3}$

36. $h(x) = \dfrac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$

$h'(x) = 2 - \dfrac{1}{x^2} = \dfrac{2x^2 - 1}{x^2}$

37. $y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$

38. $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$

$f'(x) = \dfrac{1}{3}x^{-2/3} + \dfrac{1}{5}x^{-4/5} = \dfrac{1}{3x^{2/3}} + \dfrac{1}{5x^{4/5}}$

39. $h(s) = s^{4/5}$

$$h'(s) = \frac{4}{5}s^{-1/5} = \frac{4}{5s^{1/5}}$$

40. $f(t) = t^{1/3} - 1$

$$f'(t) = \frac{1}{3}t^{-2/3} = \frac{1}{3t^{2/3}}$$

41. $f(x) = 4\sqrt{x} + 3\cos x = 4x^{1/2} + 3\cos x$

$$f'(x) = 2x^{-1/2} - 3\sin x$$

$$= \frac{2}{\sqrt{x}} - 3\sin x$$

42. $f(x) = 2\sin x + 3\cos x$

$$f'(x) = 2\cos x - 3\sin x$$

43. (a) $y = x^4 - 3x^2 + 2$

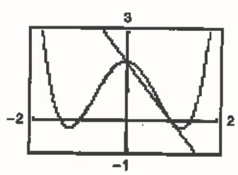
$y' = 4x^3 - 6x$

At $(1, 0)$: $y' = 4(1)^3 - 6(1) = -2.$

Tangent line: $y - 0 = -2(x - 1)$

$$2x + y - 2 = 0$$

(b)



44. (a) $y = x^3 + x$

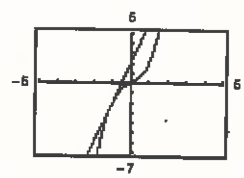
$y' = 3x^2 + 1$

At $(-1, -2)$: $y' = 3(-1)^2 + 1 = 4.$

Tangent line: $y + 2 = 4(x + 1)$

$$4x - y + 2 = 0$$

(b)



45. (a) $f(x) = \dfrac{1}{\sqrt[3]{x^2}} = x^{-2/3}$

$$f'(x) = -\frac{2}{3}x^{-5/3} = -\frac{2}{3\sqrt[3]{x^5}}$$

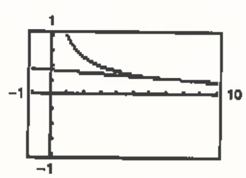
At $\left(8, \dfrac{1}{4}\right)$: $y' = -\dfrac{2}{3(\sqrt[3]{8})^5} = -\dfrac{1}{48}.$

Tangent line: $y - \dfrac{1}{4} = -\dfrac{1}{48}(x - 8)$

$$-48y + 12 = x - 8$$

$$0 = x + 48y - 20$$

(b)



46. (a) $y = (x^2 + 2x)(x + 1)$

$= x^3 + 3x^2 + 2x$

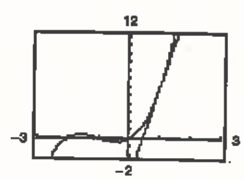
$y' = 3x^2 + 6x + 2$

At $(1, 6)$: $y' = 3(1)^2 + 6(1) + 2 = 11.$

Tangent line: $y - 6 = 11(x - 1)$

$$0 = 11x - y - 5$$

(b)



47. $y = x^4 - 8x^2 + 2$

$y' = 4x^3 - 16x$

$= 4x(x^2 - 4)$

$= 4x(x - 2)(x + 2)$

$y' = 0 \Rightarrow x = 0, \pm 2$

Horizontal tangents: $(0, 2), (2, -14), (-2, -14)$

48. $y = x^3 + x$

$y' = 3x^2 + 1 > 0$ for all x.

Therefore, there are no horizontal tangents.

49. $y = \dfrac{1}{x^2} = x^{-2}$

$y' = -2x^{-3} = \dfrac{-2}{x^3}$ cannot equal zero.

Therefore, there are no horizontal tangents.

50. $y = x^2 + 1$

$y' = 2x = 0 \Rightarrow x = 0$

At $x = 0, y = 1$.

Horizontal tangent: $(0, 1)$

51. $y = x + \sin x, 0 \le x < 2\pi$

$y' = 1 + \cos x = 0$

$\cos x = -1 \Rightarrow x = \pi$

At $x = \pi, y = \pi$.

Horizontal tangent: (π, π)

52. $y = \sqrt{3}x + 2\cos x,\ 0 \le x < 2\pi$

$y' = \sqrt{3} - 2\sin x = 0$

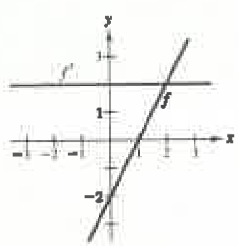
$\sin x = \dfrac{\sqrt{3}}{2} \Rightarrow x = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

At $x = \dfrac{\pi}{3}, y = \dfrac{\sqrt{3}\pi + 3}{3}$.

At $x = \dfrac{2\pi}{3}, y = \dfrac{2\sqrt{3}\pi - 3}{3}$.

Horizontal tangents: $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}\pi + 3}{3}\right), \left(\dfrac{2\pi}{3}, \dfrac{2\sqrt{3}\pi - 3}{3}\right)$

53.

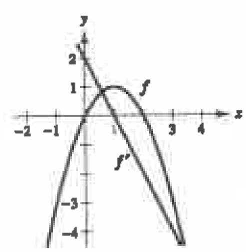


If f is linear then its derivative is a constant function.

$f(x) = ax + b$

$f'(x) = a$

54.



If f is quadratic its derivative is a linear function.

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

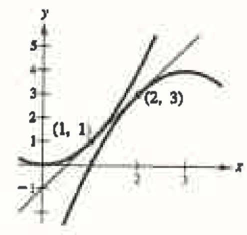
55. Let (x_1, y_1) and (x_2, y_2) be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively. The derivatives of these functions are

$$y' = 2x \Longrightarrow m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \Longrightarrow m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$,



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (-x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

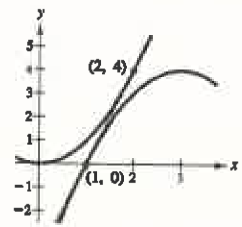
$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$



$x_2 = 1 \Longrightarrow y_2 = 0, x_1 = 2$ and $y_1 = 4$

Thus, the tangent line through $(1, 0)$ and $(2, 4)$ is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Longrightarrow y = 4x - 4.$$

$x_2 = 2 \Longrightarrow y_2 = 3, x_1 = 1$ and $y_1 = 1$

Thus, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Longrightarrow y = 2x - 1.$$

56. m_1 is the slope of the line tangent to $y = x$. m_2 is the slope of the line tangent to $y = 1/x$. Since

$$y = x \Longrightarrow y' = 1 \Longrightarrow m_1 = 1 \text{ and } y = \frac{1}{x} \Longrightarrow y' = \frac{-1}{x^2} \Longrightarrow m_2 = \frac{-1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \Longrightarrow x^2 = 1 \Longrightarrow x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular at the points intersection.

57. $f(x) = \sqrt{x}$, $(-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0-y}{-4-x}$$

$$4 + x = 2\sqrt{x}y$$

$$4 + x = 2\sqrt{x}\sqrt{x}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

The point $(4, 2)$ is on the graph of f.

Tangent line: $y - 2 = \frac{0-2}{-4-4}(x-4)$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

58. $f(x) = \frac{2}{x}$, $(5, 0)$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0-y}{5-x}$$

$$-10 + 2x = -x^2y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

$$x = \frac{5}{2}, y = \frac{4}{5}$$

The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of f. The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$

$$25y - 20 = -8x + 20$$

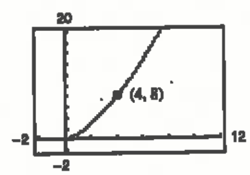
$$8x + 25y - 40 = 0$$

59. (a) One possible secant is between $(3.9, 7.7019)$ and $(4, 8)$:

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

$$y = S(x) = 2.981x - 3.924$$

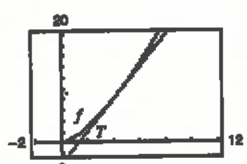


(b) $f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$

$$T(x) = 3(x - 4) + 8 = 3x - 4$$

$S(x)$ is an approximation of the tangent line $T(x)$.

(c) As you move further away from $(4, 8)$, the accuracy of the approximation T gets worse.



(d)

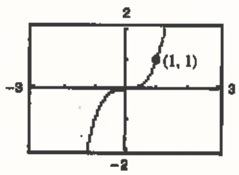
Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(x)$	-1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

60. (a) Nearby point: $(1.0073138, 1.0221024)$

Secant line: $y - 1 = \dfrac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$

$$y = 3.022(x - 1) + 1$$

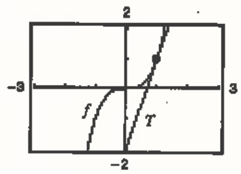
(Answers will vary.)



(b) $f'(x) = 3x^2$

$T(x) = 3(x - 1) + 1 = 3x - 2$

(c) The accuracy worsens at you move away from $(1, 1)$.



(d)

Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(x)$	-8	-1	0	0.125	0.729	1	1.331	3.375	8	27	64
$T(x)$	-8	-5	-2	-0.5	0.7	1	1.3	2.5	4	7	10

The accuracy decreases more rapidly than in Exercise 59 because $y = x^3$ is less "linear" than $y = x^{3/2}$.

61. False. Let $f(x) = x^2$ and $g(x) = x^2 + 4$. Then $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.

62. True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

63. False. If $y = \pi^2$, then $dy/dx = 0$. (π^2 is a constant.)

64. True. If $y = x/\pi = (1/\pi) \cdot x$, then $dy/dx = 1/\pi(1) = 1/\pi$.

65. $f(t) = 2t + 7, [1, 2]$

$f'(t) = 2$

Instantaneous rate of change is the constant 2.
Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because f is a line of slope 2.)

66. $f(t) = t^2 - 3, [2, 2.1]$

$f'(t) = 2t$

Instantaneous rate of change:

$(2, 1) \Rightarrow f'(2) = 2(2) = 4$

$(2.1, 1.41) \Rightarrow f'(2.1) = 4.2$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

67. $f(x) = -\dfrac{1}{x}, [1, 2]$

$f'(x) = \dfrac{1}{x^2}$

Instantaneous rate of change:

$(1, -1) \Rightarrow f'(1) = 1$

$\left(2, -\dfrac{1}{2}\right) \Rightarrow f'(2) = \dfrac{1}{4}$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

68. $f(x) = \sin x, \left[0, \dfrac{\pi}{6}\right]$

$f'(x) = \cos x$

Instantaneous rate of change:

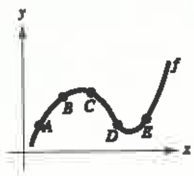
$(0, 0) \Rightarrow f'(0) = 1$

$\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right) \Rightarrow f'\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} \approx 0.866$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

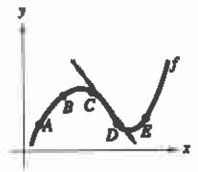
69.



(a) The slope appears to be steepest between A and B.

(b) The average rate of change between A and B is greater than the instantaneous rate of change at B.

(c)



(d) The average rates of change are approximately equal between B and C, and between D and E.

70. The graph of a function f such that $f' > 0$ for all x and the rate of change the function is decreasing (i.e. $f'' < 0$)would, in general, look like the graph at the right.

71. (a) $s(t) = -16t^2 + 1362$

$v(t) = -32t$

(b) $\dfrac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48$ ft/sec

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32$ ft/sec.

When $t = 2$: $v(2) = -64$ ft/sec.

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $$v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right)$$

$$= -8\sqrt{1362} \approx -295.242 \text{ ft/sec}$$

72.

$s(t) = -16t^2 - 22t + 220$

$v(t) = -32t - 22$

$v(3) = -118$ ft/sec

$s(t) = -16t^2 - 22t + 220$

$= 112$ (height after falling 108 ft)

$-16t^2 - 22t + 108 = 0$

$-2(t - 2)(8t + 27) = 0$

$t = 2$

$v(2) = -32(2) - 22$

$= -86$ ft/sec

73. $s(t) = -4.9t^2 + v_0t + s_0$

$= -4.9t^2 + 120t$

$v(t) = -9.8t + 120$

$v(5) = -9.8(5) + 120 = 71$ m/sec

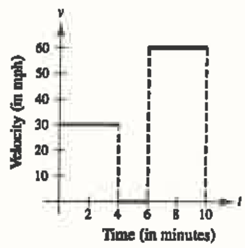
$v(10) = -9.8(10) + 120 = 22$ m/sec

74. $s(t) = -4.9t^2 + v_0t + s_0$

$= -4.9t^2 + s_0 = 0$ when $t = 6.8$.

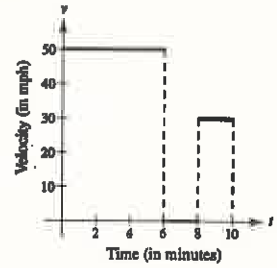
$s_0 = 4.9t^2 = 4.9(6.8)^2 = 226.6$ m

75.



(The velocity has been converted to miles per hour)

76.



(The velocity has been converted to miles per hour)

77. $v = 40 \text{ mph} = \frac{2}{3} \text{ mi/min}$

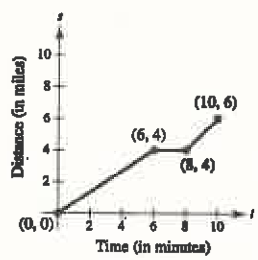
$\left(\frac{2}{3} \text{ mi/min}\right)(6 \text{ min}) = 4 \text{ mi}$

$v = 0 \text{ mph} = 0 \text{ mi/min}$

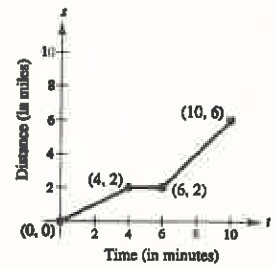
$(0 \text{ mi/min})(2 \text{ min}) = 0 \text{ mi}$

$v = 60 \text{ mph} = 1 \text{mi/min}$

$(1 \text{ mi/min})(2 \text{ min}) = 2 \text{ mi}$



78. This graph corresponds with Exercise 75.



79. (a) Using a graphing utility, you obtain

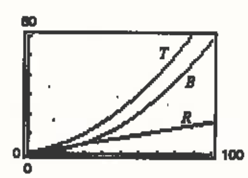
$$R = 0.167v - 0.02.$$

(b) Using a graphing utility, you obtain

$$B = 0.00586v^2 - 0.0239v + 0.46.$$

(c) $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(d)



(e) $\dfrac{dT}{dv} = 0.01172v + 0.1431$

For $v = 40$, $T'(40) \approx 0.612$.

For $v = 80$, $T'(80) \approx 1.081$.

For $v = 100$, $T'(100) \approx 1.315$.

(f) For increasing speeds, the total stopping distance increases.

80. $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$.

$$\text{Average velocity: } \frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} = \frac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c)]}{2\Delta t}$$

$$= \frac{-(1/2)a(t_0^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a(t_0^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t}$$

$$= \frac{-2at_0\Delta t}{2\Delta t}$$

$$= -at_0$$

$$= s'(t_0) \quad \text{Instantaneous velocity at } t = t_0$$

81. $A = s^2, \dfrac{dA}{ds} = 2s$

When $s = 4$ m, $\dfrac{dA}{ds} = 8 \text{ m}^2$.

82. $V = s^3, \dfrac{dV}{ds} = 3s^2$

When $s = 4$ cm, $\dfrac{dV}{ds} = 48 \text{ cm}^3$.

83. $$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

$$\frac{dC}{dQ} = -\frac{1{,}008{,}000}{Q^2} + 6.3$$

$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$

When $Q = 350$, $\frac{dC}{dQ} \approx -\$1.93$.

84. $C =$ (gallons of fuel used)(cost per gallon)

$$= \left(\frac{15{,}000}{x}\right)(1.25) = \frac{18{,}750}{x}$$

$$\frac{dC}{dx} = -\frac{18{,}750}{x^2}$$

x	10	15	20	25	30	35	40
C	1875	1250	537.5	750	625	535.71	468.75
$\frac{dC}{dx}$	-187.5	-83.333	-46.875	-30	-20.833	-15.306	-11.719

The driver who gets 15 miles per gallon would benefit more from a 1 mile per gallon increase in fuel efficiency. The rate of change is larger when $x = 15$.

85. The runner was safe. Explanations will vary.

86. $\frac{dT}{dt} = K(T - T_a)$

87. $y = ax^2 + bx + c$

Since the parabola passes through $(0, 1)$ and $(1, 0)$, we have

$(0, 1)$: $1 = a(0)^2 + b(0) + c \implies c = 1$

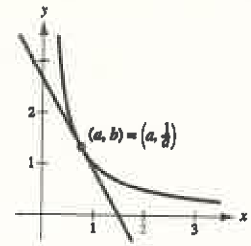
$(1, 0)$: $0 = a(1)^2 + b(1) + 1 \implies b = -a - 1.$

Thus, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point $(1, 0)$.

$$y' = 2ax + (-a - 1)$$
$$1 = 2a(1) + (-a - 1)$$
$$1 = a - 1$$
$$a = 2$$
$$b = -a - 1 = -3$$

Therefore, $y = 2x^2 - 3x + 1$.

88. $y = \frac{1}{x},\ x > 0$



$y' = -\frac{1}{x^2}$

At (a, b), the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}.$$

The x-intercept is $(2a, 0)$.

The y-intercept is $\left(0, \frac{2}{a}\right)$.

The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$.

89. $y = x^3 - 9x$

$y' = 3x^2 - 9$

Tangent lines through $(1, -9)$:

$$\begin{aligned} y + 9 &= (3x^2 - 9)(x - 1) \\ (x^3 - 9x) + 9 &= 3x^3 - 3x^2 - 9x + 9 \\ 0 &= 2x^3 - 3x^2 = x^2(2x - 3) \\ x &= 0 \text{ or } x = \tfrac{3}{2} \end{aligned}$$

The points of tangency are $(0, 0)$ and $\left(\frac{3}{2}, -\frac{81}{8}\right)$. At $(0, 0)$ the slope is $y'(0) = -9$. At $\left(\frac{3}{2}, -\frac{81}{8}\right)$ the slope is $y'\left(\frac{3}{2}\right) = -\frac{9}{4}$.

Tangent lines:

$$\begin{aligned} y - 0 &= -9(x - 0) \quad \text{and} \quad & y + \tfrac{81}{8} &= -\tfrac{9}{4}\left(x - \tfrac{3}{2}\right) \\ y &= -9x & y &= -\tfrac{9}{4}x - \tfrac{27}{4} \\ 9x + y &= 0 & 9x + 4y + 27 &= 0 \end{aligned}$$

90. $y = x^2$

$y' = 2x$

(a) Tangent lines through $(0, a)$:

$$\begin{aligned} y - a &= 2x(x - 0) \\ x^2 - a &= 2x^2 \\ -a &= x^2 \\ \pm\sqrt{-a} &= x \end{aligned}$$

The points of tangency are $(\pm\sqrt{-a}, -a)$. At $(\sqrt{-a}, -a)$ the slope is $y'(\sqrt{-a}) = 2\sqrt{-a}$. At $(-\sqrt{-a}, -a)$ the slope is $y'(-\sqrt{-a}) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}(x - \sqrt{-a})$ and $y + a = -2\sqrt{-a}(x + \sqrt{-a})$

$y = 2\sqrt{-a}\,x + a \qquad\qquad y = -2\sqrt{-a}\,x + a$

Restriction: a must be negative.

—CONTINUED—

90. —CONTINUED—

(b) Tangent lines through $(a, 0)$:

$$y - 0 = 2x(x - a)$$
$$x^2 = 2x^2 - 2ax$$
$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$ the slope is $y'(0) = 0$. At $(2a, 4a^2)$ the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$y = 0$ $\qquad$ $y = 4ax - 4a^2$

Restriction: None, a can be any real number.

91. $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

f must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} ax^3 = 8a \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (x^2 + b) = 4 + b \end{aligned}\right\} \quad \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For f to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$$3a(2)^2 = 2(2)$$
$$12a = 4$$
$$a = \tfrac{1}{3}$$
$$b = 8a - 4 = -\tfrac{4}{3}$$

92. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.

$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

You can verify this by graphing f_1 and f_2 and observing the locations of the sharp turns.

93. Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x\to 0} \sin x\left(\frac{\sin \Delta x}{\Delta x}\right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

1. $f(x) = \frac{1}{3}(2x^3 - 4)$

$f'(x) = \frac{1}{3}(6x^2) = 2x^2$

$f'(0) = 0$

2. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$\begin{aligned} f'(x) &= (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2) \\ &= 3x^2(x-1)^2 + 2(x-1)^2(x^2 + x + 1) \\ &= (x-1)^2(5x^2 + 2x + 2) \end{aligned}$$

$f'(1) = 0$

3. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$\begin{aligned} f'(x) &= (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15 \end{aligned}$$

$f'(0) = -15$

4. $f(x) = \dfrac{x+1}{x-1}$

$$\begin{aligned} f'(x) &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} \\ &= \frac{x - 1 - x - 1}{(x-1)^2} \\ &= -\frac{2}{(x-1)^2} \end{aligned}$$

$f'(2) = -\dfrac{2}{(2-1)^2} = -2$

5. $f(x) = x\cos x$

$$\begin{aligned} f'(x) &= (x)(-\sin x) + (\cos x)(1) \\ &= \cos x - x\sin x \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{8}(4 - \pi) \end{aligned}$$

6. $f(x) = \dfrac{\sin x}{x}$

$$\begin{aligned} f'(x) &= \frac{(x)(\cos x) - (\sin x)(1)}{x^2} \\ &= \frac{x\cos x - \sin x}{x^2} \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{6}\right) &= \frac{(\pi/6)(\sqrt{3}/2) - (1/2)}{\pi^2/36} \\ &= \frac{3\sqrt{3}\pi - 18}{\pi^2} \\ &= \frac{3(\sqrt{3}\pi - 6)}{\pi^2} \end{aligned}$$

Function	Rewrite	Derivative	Simplify
7. $y = \dfrac{x^2 + 2x}{x}$	$y = x + 2,\ x \neq 0$	$y' = 1$	$y' = 1$
8. $y = \dfrac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, x > 0$	$y' = 2x^{-1/2}$	$y' = \dfrac{2}{\sqrt{x}}$
9. $y = \dfrac{7}{3x^3}$	$y = \dfrac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\dfrac{7}{x^4}$
10. $y = \dfrac{4}{5x^2}$	$y = \dfrac{4}{5}x^{-2}$	$y' = -\dfrac{8}{5}x^{-3}$	$y' = -\dfrac{8}{5x^3}$
11. $y = \dfrac{3x^2 - 5}{7}$	$y = \dfrac{1}{7}(3x^2 - 5)$	$y' = \dfrac{1}{7}(6x)$	$y' = \dfrac{6x}{7}$
12. $y = \dfrac{x^2 - 4}{x + 2}$	$y = x - 2,\ x \neq -2$	$y' = 1$	$y' = 1$

13. $f(x) = \dfrac{3x - 2}{2x - 3}$

$$f'(x) = \frac{(2x - 3)(3) - (3x - 2)(2)}{(2x - 3)^2} = \frac{-5}{(2x - 3)^2}$$

14. $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2} = \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}$$

15. $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$$

$$= \frac{2}{(x + 1)^2}, x \neq 1$$

16. $f(x) = x^4\left[1 - \dfrac{2}{x + 1}\right] = x^4\left[\dfrac{x - 1}{x + 1}\right]$

$$f'(x) = x^4\left[\frac{(x + 1) - (x - 1)}{(x + 1)^2}\right] + \left[\frac{x - 1}{x + 1}\right](4x^3)$$

$$= 2x^3\left[\frac{2x^2 + x - 2}{(x + 1)^2}\right]$$

17. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

$$f'(x) = \frac{\sqrt{x}(1) - (x + 1)\left[1/\left(2\sqrt{x}\right)\right]}{x}$$

$$= \frac{x - 1}{2x^{3/2}}$$

Alternate solution:

$$f(x) = \frac{x + 1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$$

$$= \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}$$

$$= \frac{x - 1}{2x^{3/2}}$$

18. $f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right) = x^{1/3}(x^{1/2} + 3)$

$$f'(x) = x^{1/3}\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 3)\left(\frac{1}{3}x^{-2/3}\right)$$

$$= \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

Alternate solution:

$$f(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$$

$$= x^{5/6} + 3x^{1/3}$$

$$f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$$

$$= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$$

19. $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$

$$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$$

20. $h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$

$$h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

21. $h(t) = \dfrac{t + 1}{t^2 + 2t + 2}$

$$h'(t) = \frac{(t^2 + 2t + 2)(1) - (t + 1)(2t + 2)}{(t^2 + 2t + 2)^2}$$

$$= \frac{-t^2 - 2t}{(t^2 + 2t + 2)^2}$$

22. $f(x) = \dfrac{x(x^2 - 1)}{x + 3} = \dfrac{x^3 - x}{x + 3}$

$$f'(x) = \frac{(x + 3)(3x^2 - 1) - (x^3 - x)}{(x + 3)^2}$$

$$= \frac{2x^3 + 9x^2 - 3}{(x + 3)^2}$$

23. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$

$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$

$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$

$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$

$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$

24. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$

$f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1)$

$= (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1)$

$= 2x^5 + x^4 + 3x^3 + x - 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x$

$= 6x^5 + 4x^3 - 3x^2 - 1$

25. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$

$f'(x) = \dfrac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$

$= \dfrac{-4xc^2}{(x^2 - c^2)^2}$

26. $f(x) = \dfrac{c^2 - x^2}{c^2 + x^2}$

$f'(x) = \dfrac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$

$= \dfrac{-4xc^2}{(c^2 + x^2)^2}$

27. $f(t) = t^2 \sin t$

$f'(t) = t^2 \cos t + 2t \sin t$

$= t(t \cos t + 2 \sin t)$

28. $f(\theta) = (\theta + 1) \cos \theta$

$f'(\theta) = (\theta + 1)(-\sin \theta) + (\cos \theta)(1)$

$= \cos \theta - (\theta + 1) \sin \theta$

29. $f(t) = \dfrac{\cos t}{t}$

$f'(t) = \dfrac{-t \sin t - \cos t}{t^2} = -\dfrac{t \sin t + \cos t}{t^2}$

30. $f(x) = \dfrac{\sin x}{x}$

$f'(x) = \dfrac{x \cos x - \sin x}{x^2}$

31. $f(x) = -x + \tan x$

$f'(x) = -1 + \sec^2 x = \tan^2 x$

32. $y = x + \cot x$

$y' = 1 - \csc^2 x = -\cot^2 x$

33. $g(t) = \sqrt{t} + 4 \sec t$

$g'(t) = \dfrac{1}{2}t^{-1/2} + 4 \sec t \tan t$

$= \dfrac{1}{2\sqrt{t}} + 4 \sec t \tan t$

34. $h(s) = \dfrac{1}{s} - 10 \csc s$

$h'(s) = -\dfrac{1}{s^2} + 10 \csc s \cot s$

35. $y = 5x \csc x$

$y' = -5x \csc x \cot x + 5 \csc x$

$= 5 \csc x(-x \cot x + 1)$

$= 5 \csc x(1 - x \cot x)$

36. $y = \dfrac{\sec x}{x}$

$y' = \dfrac{x \sec x \tan x - \sec x}{x^2}$

$= \dfrac{\sec x(x \tan x - 1)}{x^2}$

37. $y = -\csc x - \sin x$

$$y' = \csc x \cot x - \cos x$$
$$= \frac{\cos x}{\sin^2 x} - \cos x$$
$$= \cos x(\csc^2 x - 1)$$
$$= \cos x \cot^2 x$$

38. $y = x \sin x + \cos x$

$$y' = x \cos x + \sin x - \sin x = x \cos x$$

39. $y = x^2 \sin x + 2x \cos x$

$$y' = x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x$$
$$= x^2 \cos x + 2 \cos x$$

40. $f(x) = \sin x \cos x$

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos 2x$$

41. $f(x) = x^2 \tan x$

$$f'(x) = x^2 \sec^2 x + 2x \tan x$$
$$= x(x \sec^2 x + 2 \tan x)$$

42. $h(\theta) = 5 \sec \theta + \tan \theta$

$$h'(\theta) = 5 \sec \theta \tan \theta + \sec^2 \theta$$
$$= \sec \theta(5 \tan \theta + \sec \theta)$$

43. $g(x) = \left(\frac{x+1}{x+2}\right)(2x - 5)$

$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2}$ (form of answer may vary)

44. $f(x) = \left(\frac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

$f'(x) = 2\frac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2}$ (form of answer may vary)

45. $g(\theta) = \frac{\theta}{1 - \sin \theta}$

$g'(\theta) = \frac{1 - \sin \theta + \theta \cos \theta}{(\sin \theta - 1)^2}$ (form of answer may vary)

46. $f(\theta) = \frac{\sin \theta}{1 - \cos \theta}$

$$f'(\theta) = \frac{1}{\cos \theta - 1} = \frac{\cos \theta - 1}{(1 - \cos \theta)^2}$$

(form of answer may vary)

47. $y = \frac{1 + \csc x}{1 - \csc x}$

$$y' = \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \frac{-2 \csc x \cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)(\sqrt{3})}{(1-2)^2} = -4\sqrt{3}$$

48. $f(x) = \tan x \ \cot x = 1$

$$f'(x) = 0$$
$$f'(1) = 0$$

49. $h(t) = \frac{\sec t}{t}$

$$h'(t) = \frac{t(\sec t \tan t) - (\sec t)(1)}{t^2}$$
$$= \frac{\sec t(t \tan t - 1)}{t^2}$$
$$h'(\pi) = \frac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

50. $f(x) = \sin x(\sin x + \cos x)$

$$f'(x) = \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x$$
$$= \sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x$$
$$= \sin 2x + \cos 2x$$
$$f'\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1$$

51. (a) $f(x) = \dfrac{x}{x-1}, (2, 2)$

$f'(x) = \dfrac{(x-1)(1) - x(1)}{(x-1)^2} = \dfrac{-1}{(x-1)^2}$

$f'(2) = \dfrac{-1}{(2-1)^2} = -1 = $ slope at $(2, 2)$.

Tangent line: $y - 2 = -1(x - 2) \Rightarrow y = -x + 4$

(b)

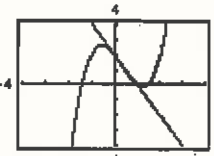
52. (a) $f(x) = (x-1)(x^2-2), \ (0, 2)$

$f'(x) = (x-1)(2x) + (x^2-2)(1) = 3x^2 - 2x - 2$

$f'(0) = -2 = $ slope at $(0, 2)$.

Tangent line: $y - 2 = -2x \Rightarrow y = -2x + 2$

(b)



53. (a) $f(x) = (x^3 - 3x + 1)(x + 2), \ (1, -3)$

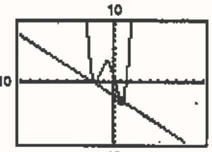
$f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$

$= 4x^3 + 6x^2 - 6x - 5$

$f'(1) = -1 = $ slope at $(1, -3)$.

Tangent line: $y + 3 = -1(x - 1) \Rightarrow y = -x - 2$

(b)



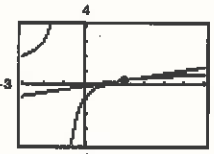
54. (a) $f(x) = \dfrac{x-1}{x+1}, \left(2, \dfrac{1}{3}\right)$

$f'(x) = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$

$f'(2) = \dfrac{2}{9} = $ slope at $\left(2, \dfrac{1}{3}\right)$.

Tangent line: $y - \dfrac{1}{3} = \dfrac{2}{9}(x - 2) \Rightarrow y = \dfrac{2}{9}x - \dfrac{1}{9}$

(b)



55. (a) $f(x) = \tan x, \ \left(\dfrac{\pi}{4}, 1\right)$

$f'(x) = \sec^2 x$

$f'\left(\dfrac{\pi}{4}\right) = 2 = $ slope at $\left(\dfrac{\pi}{4}, 1\right)$.

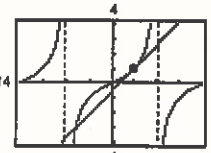
Tangent line:

$y - 1 = 2\left(x - \dfrac{\pi}{4}\right)$

$y - 1 = 2x - \dfrac{\pi}{2}$

$4x - 2y - \pi + 2 = 0$

(b)



56. (a) $f(x) = \sec x, \left(\frac{\pi}{3}, 2\right)$

$f'(x) = \sec x \tan x$

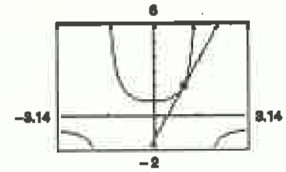
$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3} = \text{slope at } \left(\frac{\pi}{3}, 2\right).$

Tangent line:

$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$

$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$

(b)



57. $f(x) = \frac{x^2}{x-1}$

$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$

$= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$

$f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

58. $f(x) = \frac{x^2}{x^2+1}$

$f'(x) = \frac{(x^2+1)(2x) - (x^2)(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$

$f'(x) = 0$ when $x = 0$.

Horizontal tangent is at $(0, 0)$.

59. $f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + 4$

$= 0$

60. $f(x) = 4 - h(x)$

$f'(x) = -h'(x)$

$f'(2) = -h'(2) = -4$

61. $f(x) = \frac{g(x)}{h(x)}$

$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$

$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$

$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$

$= -10$

62. $f(x) = g(x)h(x)$

$f'(x) = g(x)h'(x) + h(x)g'(x)$

$f'(2) = g(2)h'(2) + h(2)g'(2)$

$= (3)(4) + (-1)(-2)$

$= 14$

63. $f(x) = x^n \sin x$

$f'(x) = x^n \cos x + nx^{n-1} \sin x$

$= x^{n-1}(x \cos x + n \sin x)$

When $n = 1$: $f'(x) = x \cos x + \sin x$.

When $n = 2$: $f'(x) = x(x \cos x + 2 \sin x)$.

When $n = 3$: $f'(x) = x^2(x \cos x + 3 \sin x)$.

When $n = 4$: $f'(x) = x^3(x \cos x + 4 \sin x)$.

For general n, $f'(x) = x^{n-1}(x \cos x + n \sin x)$.

64. $f(x) = \dfrac{\cos x}{x^n} = x^{-n}\cos x$

$$f'(x) = -x^{-n}\sin x - nx^{-n-1}\cos x$$
$$= -x^{-n-1}(x\sin x + n\cos x)$$
$$= -\frac{x\sin x + n\cos x}{x^{n+1}}$$

When $n = 1$: $f'(x) = -\dfrac{x\sin x + \cos x}{x^2}$.

When $n = 2$: $f'(x) = -\dfrac{x\sin x + 2\cos x}{x^3}$.

When $n = 3$: $f'(x) = -\dfrac{x\sin x + 3\cos x}{x^4}$.

When $n = 4$: $f'(x) = -\dfrac{x\sin x + 4\cos x}{x^5}$.

For general n, $f'(x) = -\dfrac{x\sin x + n\cos x}{x^{n+1}}$.

65. $C = 100\left(\dfrac{200}{x^2} + \dfrac{x}{x+30}\right)$, $1 \le x$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

(a) When $x = 10$: $\dfrac{dC}{dx} = -\$38.13$.

(b) When $x = 15$: $\dfrac{dC}{dx} = -\$10.37$.

(c) When $x = 20$: $\dfrac{dC}{dx} = -\$3.80$.

As the order size increases, the cost per item decreases.

66. $P = \dfrac{k}{V}$

$$\frac{dP}{dV} = -\frac{k}{V^2}$$

67. $P(t) = 500\left[1 + \dfrac{4t}{50 + t^2}\right]$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right] = 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right] = 2000\left[\frac{50 - t^2}{(50 + t^2)^2}\right]$$

$P'(2) \approx 31.55$ per hour

68. $f(x) = \sec x$

$g(x) = \csc x$, $[0, 2\pi)$

$f'(x) = g'(x)$

$$\sec x\tan x = -\csc x\cot x \Longrightarrow \frac{\sec x\tan x}{\csc x\cot x} = -1 \Longrightarrow \frac{\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}}{\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x}} = -1 \Longrightarrow$$

$$\frac{\sin^3 x}{\cos^3 x} = -1 \Longrightarrow \tan^3 x = -1 \Longrightarrow \tan x = -1$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

69. (a) $\sec x = \dfrac{1}{\cos x}$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

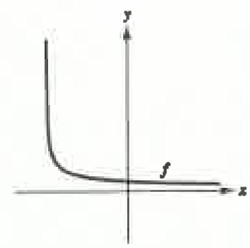
(b) $\csc x = \dfrac{1}{\sin x}$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

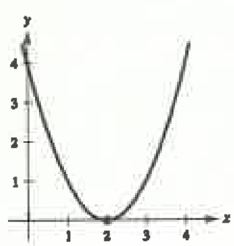
(c) $\cot x = \dfrac{\cos x}{\sin x}$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

70. The graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x would in general look like the graph at the right.



71.



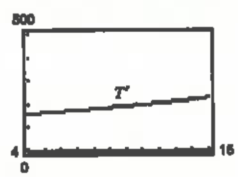
$f(2) = 0$

One such function is $f(x) = (x - 2)^2$.

72. $T = \dfrac{2{,}231{,}291 + 110{,}636t}{1000 - 14t}$

(a) Using a symbolic differentiation utility, or the quotient rule,

$$T' = \frac{70{,}937{,}037}{2(7t - 500)^2}.$$



(b) The graph of T' is getting more and more positive. That means that the tax burden per capita is increasing at a greater and greater rate.

(c) $T(20) \approx 6172$ dollars

73. $f(x) = 4x^{3/2}$

$f'(x) = 6x^{1/2}$

$f''(x) = 3x^{-1/2} = \dfrac{3}{\sqrt{x}}$

74. $f(x) = \dfrac{x^2 + 2x - 1}{x} = x + 2 - \dfrac{1}{x}$

$f'(x) = 1 + \dfrac{1}{x^2}$

$f''(x) = -\dfrac{2}{x^3}$

75. $f(x) = \dfrac{x}{x-1}$

$f'(x) = \dfrac{(x-1)(1) - x(1)}{(x-1)^2} = \dfrac{-1}{(x-1)^2}$

$f''(x) = \dfrac{2}{(x-1)^3}$

76. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3}$

$f''(x) = \dfrac{192}{x^4}$

77. $f(x) = 3\sin x$

$f'(x) = 3\cos x$

$f''(x) = -3\sin x$

78. $f(x) = \sec x$

$f'(x) = \sec x \tan x$

$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$

$= \sec x(\sec^2 x + \tan^2 x)$

79. $f'(x) = x^2$

$f''(x) = 2x$

80. $f''(x) = 2 - 2x^{-1}$

$f'''(x) = 2x^{-2} = \dfrac{2}{x^2}$

81. $f'''(x) = 2\sqrt{x}$

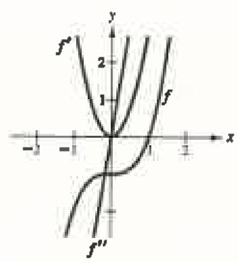
$f^{(4)}(x) = \dfrac{1}{2}(2)x^{-1/2} = \dfrac{1}{\sqrt{x}}$

82. $f^{(4)}(x) = 2x + 1$

$f^{(5)}(x) = 2$

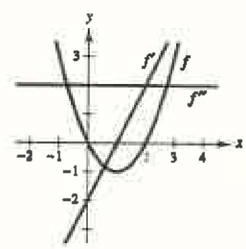
$f^{(6)}(x) = 0$

83.



It appears that f is cubic; so f' would be quadratic and f'' would be linear.

84.



It appears that f is quadratic; so f' would be linear and f'' would be constant.

85. $f(x) = g(x)h(x)$

(a) $f'(x) = g(x)h'(x) + h(x)g'(x)$

$f''(x) = g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)$

$= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)$

$f'''(x) = g(x)h'''(x) + g'(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x)$

$= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$

$f^{(4)}(x) = g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g'(x)h'''(x) + 3g''(x)h''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x)$

$+ g'''(x)h'(x) + g^{(4)}(x)h(x)$

$= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$

—CONTINUED—

85. —CONTINUED—

(b) $$f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{1[(n-1)(n-2)\cdots(2)(1)]}g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{(2)(1)[(n-2)(n-3)\cdots(2)(1)]}g''(x)h^{(n-2)}(x)$$

$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{(3)(2)(1)[(n-3)(n-4)\cdots(2)(1)]}g'''(x)h^{(n-3)}(x) + \cdots$$

$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{[(n-1)(n-2)\cdots(2)(1)](1)}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

$$= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots$$

$$+ \frac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

Note: $n! = n(n-1)\ldots 3\cdot 2\cdot 1$ (read "n factorial.")

86. (a) $f(x) = x^n$

$f^n(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

Note: $n! = n(n-1)\cdots 3\cdot 2\cdot 1$ (read "n factorial.")

(b) $f(x) = \dfrac{1}{x}$

$$f^{(n)}(x) = \frac{(-1)^n(n)(n-1)(n-2)\cdots(2)(1)}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}$$

87. $v(t) = 36 - t^2,\ 0 \le t \le 6$

$a(t) = -2t$

$v(3) = 27$ m/sec

$a(3) = -6$ m/sec

The speed of the object is decreasing, but the rate of the decrease is increasing.

88. $v(t) = \dfrac{100t}{2t+15}$

$$a(t) = \frac{(2t+15)(100) - (100t)(2)}{(2t+15)^2}$$

$$= \frac{1500}{(2t+15)^2}$$

(a) $a(5) = \dfrac{1500}{[2(5)+15]^2} = 2.4 \text{ ft/sec}^2$

(b) $a(10) = \dfrac{1500}{[2(10)+15]^2} \approx 1.2 \text{ ft/sec}^2$

(c) $a(20) = \dfrac{1500}{[2(20)+15]^2} \approx 0.5 \text{ ft/sec}^2$

89. $s(t) = -8.25t^2 + 66t$

$v(t) = -16.50t + 66$

$a(t) = -16.50$

t(sec)	0	1	2	3	4
$s(t)$ (ft)	0	57.75	99	123.75	132
$v(t) = s'(t)$ (ft/sec)	66	49.5	33	16.5	0
$a(t) = v'(t)$ (ft/sec^2)	−16.5	−16.5	−16.5	−16.5	−16.5

Average velocity on:

$[0, 1]$ is $\dfrac{57.75 - 0}{1 - 0} = 57.75.$

$[1, 2]$ is $\dfrac{99 - 57.75}{2 - 1} = 41.25.$

$[2, 3]$ is $\dfrac{123.75 - 99}{3 - 2} = 24.75.$

$[3, 4]$ is $\dfrac{132 - 123.75}{4 - 3} = 8.25.$

90. $s = -\frac{27}{10}t^2 + 27t + 6$

(a) $v(t) = -\frac{27}{5}t + 27$ ft/sec

$a(t) = -\frac{27}{5} = -5.4$ ft/sec^2

(b) $-\frac{27}{5}t + 27 = 0$ when $t = 5$ seconds.

$s(5) = 73.5$ feet

(c) On earth, $a = -32$ ft/sec^2.

91. $f(x) = \cos x$ $\qquad f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

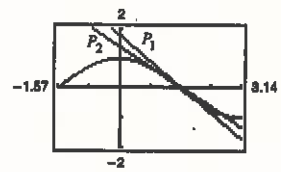
$f'(x) = -\sin x$ $\qquad f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

$f''(x) = -\cos x$ $\qquad f''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

(a) $P_1(x) = f'(a)(x-a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

$P_2(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a)$

$= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

(b) 

(c) P_2 is a better approximation.

(d) The accuracy worsens as you move farther away from $x = a = (\pi/3)$.

92. $f(x) = \tan x$ $\qquad f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ $\qquad f'\left(\frac{\pi}{4}\right) = \left(\frac{2}{\sqrt{2}}\right)^2 = 2$

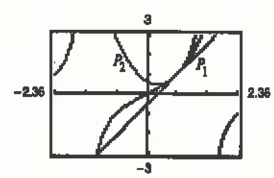
$f''(x) = 2\sec x(\sec x \tan x)$ $\qquad f''\left(\frac{\pi}{4}\right) = 2(2)(1) = 4$

(a) $P_1(x) = f'(a)(x-a) + f(a) = 2\left(x - \frac{\pi}{4}\right) + 1$

$P_2(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a)$

$= \frac{1}{2}(4)\left(x - \frac{\pi}{4}\right)^2 + 2\left(x - \frac{\pi}{4}\right) + 1$

$= 2\left(x - \frac{\pi}{4}\right)^2 + 2\left(x - \frac{\pi}{4}\right) + 1$

(b) 

(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move farther away from $x = a = (\pi/4)$.

93. False. If $y = f(x)g(x)$, then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

94. True. y is a fourth-degree polynomial.

$$\frac{d^ny}{dx^n} = 0 \text{ when } n > 4.$$

95. True

$$h'(c) = f(c)g'(c) + g(c)f'(c) = f(c)(0) + g(c)(0) = 0$$

96. True

97. True

98. True. If $v(t) = c$ then

$$a(t) = v'(t) = 0.$$

99. $f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$f'(x) = \left\{\begin{matrix} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{matrix}\right\} = 2|x|$

$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$

$f''(0)$ does not exist since the left and right derivatives are not equal.

100. (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$
$= fg'' - f''g$ True

(b) $(fg)'' = (fg' + f'g)'$
$= fg'' + f'g' + f'g' + f''g$
$= fg'' + 2f'g' + f''g$
$\neq fg'' + f''g$ False

Section 2.4 The Chain Rule

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (6x - 5)^4$	$u = 6x - 5$	$y = u^4$
2. $y = \dfrac{1}{\sqrt{x+1}}$	$u = x + 1$	$y = u^{-1/2}$
3. $y = \sqrt{x^2 - 1}$	$u = x^2 - 1$	$y = \sqrt{u}$
4. $y = \tan(\pi x + 1)$	$u = \pi x + 1$	$y = \tan u$
5. $y = \csc^3 x$	$u = \csc x$	$y = u^3$
6. $y = \cos\dfrac{3x}{2}$	$u = \dfrac{3x}{2}$	$y = \cos u$

7. $y = (2x - 7)^3$

$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$

8. $y = (3x^2 + 1)^4$

$y' = 4(3x^2 + 1)^3(6x) = 24x(3x^2 + 1)^3$

9. $g(x) = 3(4 - 9x)^4$

$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$

10. $f(x) = 2(1 - x^2)^3$

$f'(x) = 6(1 - x^2)^2(-2x) = -12x(1 - x^2)^2$

11. $f(x) = (9 - x^2)^{2/3}$

$f'(x) = \dfrac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\dfrac{4x}{3(9 - x^2)^{1/3}}$

12. $f(t) = (9t + 2)^{2/3}$

$f'(t) = \dfrac{2}{3}(9t + 2)^{-1/3}(9) = \dfrac{6}{\sqrt[3]{9t + 2}}$

13. $f(t) = (1 - t)^{1/2}$

$f'(t) = \dfrac{1}{2}(1 - t)^{-1/2}(-1) = -\dfrac{1}{2\sqrt{1 - t}}$

14. $g(x) = (3 - 2x)^{1/2}$

$g'(x) = \dfrac{1}{2}(3 - 2x)^{-1/2}(-2) = -\dfrac{1}{\sqrt{3 - 2x}}$

15. $y = (9x^2 + 4)^{1/3}$

$y' = \dfrac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \dfrac{6x}{(9x^2 + 4)^{2/3}}$

16. $g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = |x - 1|$

$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$

17. $y = 2(4 - x^2)^{1/2}$

$$y' = (4 - x^2)^{-1/2}(-2x) = -\frac{2x}{\sqrt{4 - x^2}}$$

18. $f(x) = -3(2 - 9x)^{1/4}$

$$f'(x) = -\frac{3}{4}(2 - 9x)^{-3/4}(-9) = \frac{27}{4(2 - 9x)^{3/4}}$$

19. $y = (x - 2)^{-1}$

$$y' = -1(x - 2)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

20. $s(t) = (t^2 + 3t - 1)^{-1}$

$$s'(t) = -1(t^2 + 3t - 1)^{-2}(2t + 3)$$
$$= \frac{-(2t + 3)}{(t^2 + 3t - 1)^2}$$

21. $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

22. $y = -4(t + 2)^{-2}$

$$y' = 8(t + 2)^{-3} = \frac{8}{(t + 2)^3}$$

23. $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

24. $g(t) = (t^2 - 2)^{-1/2}$

$$g'(t) = -\frac{1}{2}(t^2 - 2)^{-3/2}(2t) = -\frac{t}{(t^2 - 2)^{3/2}}$$

25. $f(x) = x^2(x - 2)^4$

$$f'(x) = x^2[4(x - 2)^3(1)] + (x - 2)^4(2x)$$
$$= 2x(x - 2)^3[2x + (x - 2)]$$
$$= 2x(x - 2)^3(3x - 2)$$

26. $f(x) = x(3x - 9)^3$

$$f'(x) = x[3(3x - 9)^2(3)] + (3x - 9)^3(1)$$
$$= (3x - 9)^2[9x + 3x - 9]$$
$$= 27(x - 3)^2(4x - 3)$$

27. $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$y' = x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1)$$
$$= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2}$$
$$= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)]$$
$$= \frac{1 - 2x^2}{\sqrt{1 - x^2}}$$

28. $y = x^2\sqrt{9 - x^2} = x^2(9 - x^2)^{1/2}$

$$y' = x^2\left[\frac{1}{2}(9 - x^2)^{-1/2}(-2x)\right] + (9 - x^2)^{1/2}(2x)$$
$$= -x^3(9 - x^2)^{-1/2} + 2x(9 - x^2)^{1/2}$$
$$= x(9 - x^2)^{-1/2}[-x^2 + 2(9 - x^2)]$$
$$= \frac{x(18 - 3x^2)}{\sqrt{9 - x^2}}$$
$$= \frac{3x(6 - x^2)}{\sqrt{9 - x^2}}$$

29. $y = \dfrac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$y' = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}(1)$$
$$= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2}$$
$$= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)]$$
$$= \frac{1}{(x^2 + 1)^{3/2}}$$

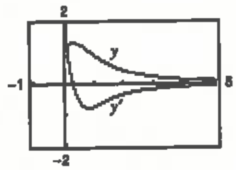
30. $y = \dfrac{x^2}{(x^9 + 9)^{1/2}}$

$$y' = \frac{(x^9 + 9)^{1/2}(2x) - x^2(1/2)(x^9 + 9)^{-1/2}\,9x^8}{x^9 + 9}$$
$$= \frac{(x^9 + 9)^{-1/2}x[2(x^9 + 9) - (9/2)x^9]}{x^9 + 9}$$
$$= \frac{x[18 - (5/2)x^9]}{(x^9 + 9)^{3/2}} = \frac{x(36 - 5x^9)}{2(x^9 + 9)^{3/2}}$$

31. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

$y' = \dfrac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$

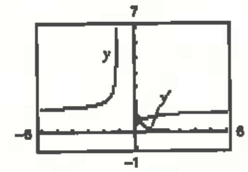
The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



32. $y = \sqrt{\dfrac{2x}{x + 1}}$

$y' = \dfrac{1}{\sqrt{2x}(x + 1)^{3/2}}$

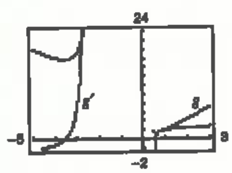
y' has no zeros.



33. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$g'(t) = \dfrac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$

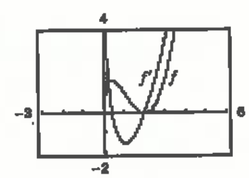
The zeros of g' correspond to the points on the graph of g where the tangent lines are horizontal.



34. $f(x) = \sqrt{x}(2 - x)^2$

$f'(x) = \dfrac{(x - 2)(5x - 2)}{2\sqrt{x}}$

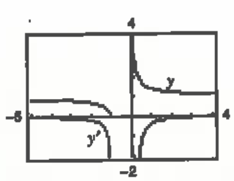
The zeros of f' correspond to the points on the graph of f where the tangent lines are horizontal.



35. $y = \sqrt{\dfrac{x + 1}{x}}$

$y' = -\dfrac{\sqrt{(x + 1)/x}}{2x(x + 1)}$

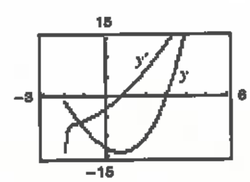
y' has no zeros.



36. $y = (t^2 - 9)\sqrt{t + 2}$

$y' = \dfrac{5t^2 + 8t - 9}{2\sqrt{t + 2}}$

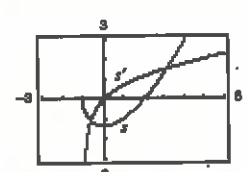
The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



37. $s(t) = \dfrac{-2(2 - t)\sqrt{1 + t}}{3}$

$s'(t) = \dfrac{t}{\sqrt{1 + t}}$

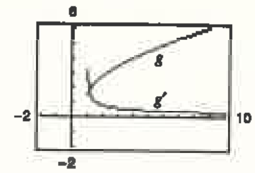
The zero of $s'(t)$ corresponds to the point on the graph of $s(t)$ where the tangent line is horizontal.



38. $g(x) = \sqrt{x-1} + \sqrt{x+1}$

$$g'(x) = \frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x+1}}$$

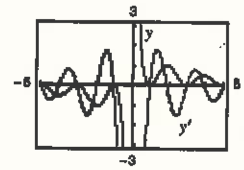
g' has no zeros.



39. $y = \dfrac{\cos \pi x + 1}{x}$

$$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$$

$$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$$

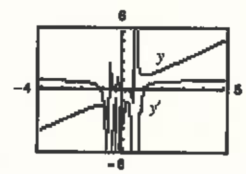


The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

40. $y = x^2 \tan \dfrac{1}{x}$

$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.



41. (a) $y = \sin x$

$y' = \cos x$

$y'(0) = 1$

1 cycle in $[0, 2\pi]$

(b) $y = \sin 2x$

$y' = 2 \cos 2x$

$y'(0) = 2$

2 cycles in $[0, 2\pi]$

42. (a) $y = \sin 3x$

$y' = 3 \cos 3x$

$y'(0) = 3$

3 cycles in $[0, 2\pi]$

(b) $y = \sin\left(\dfrac{x}{2}\right)$

$y' = \left(\dfrac{1}{2}\right) \cos\left(\dfrac{x}{2}\right)$

$y'(0) = \dfrac{1}{2}$

Half cycle in $[0, 2\pi]$

43. $y = \cos 3x$

$$\frac{dy}{dx} = -3 \sin 3x$$

44. $y = \sin \pi x$

$$\frac{dy}{dx} = \pi \cos \pi x$$

45. $g(x) = 3\tan 4x$

$g'(x) = 12\sec^2 4x$

46. $h(x) = \sec(x^2)$

$h'(x) = 2x\sec(x^2)\tan(x^2)$

47. $f(\theta) = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$f'(\theta) = 2(\frac{1}{4})(\sin 2\theta)(\cos 2\theta)(2)$

$= \sin 2\theta \cos 2\theta = \frac{1}{2}\sin 4\theta$

48. $g(t) = 5\cos^2 \pi t = 5(\cos \pi t)^2$

$g'(t) = 10\cos \pi t(-\sin \pi t)(\pi)$

$= -10\pi(\sin \pi t)(\cos \pi t) = -5\pi \sin 2\pi t$

49. $y = \sqrt{x} + \dfrac{1}{4}\sin(2x)^2$

$= \sqrt{x} + \dfrac{1}{4}\sin(4x^2)$

$\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} + \dfrac{1}{4}\cos(4x^2)(8x)$

$= \dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

50. $y = 3x - 5\cos(\pi x)^2$

$= 3x - 5\cos(\pi^2 x^2)$

$\dfrac{dy}{dx} = 3 + 5\sin(\pi^2 x^2)(2\pi^2 x)$

$= 3 + 10\pi^2 x \sin(\pi x)^2$

51. $y = \sin(\cos x)$

$\dfrac{dy}{dx} = \cos(\cos x) \cdot (-\sin x)$

$= -\sin x \cos(\cos x)$

52. $y = \sin\sqrt{x} + \sqrt{\sin x}$

$\dfrac{dy}{dx} = \cos\sqrt{x} \cdot \dfrac{1}{2}x^{-1/2} + \dfrac{1}{2}(\sin x)^{-1/2} \cdot \cos x$

$= \dfrac{\cos\sqrt{x}}{2\sqrt{x}} + \dfrac{\cos x}{2\sqrt{\sin x}}$

53. $s(t) = (t^2 + 2t + 8)^{1/2}$, $(2, 4)$

$s'(t) = \dfrac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$

$= \dfrac{t + 1}{\sqrt{t^2 + 2t + 8}}$

$s'(2) = \dfrac{3}{4}$

54. $y = (3x^3 + 4x)^{1/5}$, $(2, 2)$

$y' = \dfrac{1}{5}(3x^3 + 4x)^{-4/5}(9x^2 + 4)$

$= \dfrac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}}$

$y'(2) = \dfrac{1}{2}$

55. $f(x) = \dfrac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}$, $\left(-1, -\dfrac{3}{5}\right)$

$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\dfrac{9x^2}{(x^3 - 4)^2}$

$f'(-1) = -\dfrac{9}{25}$

56. $f(x) = \dfrac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}$, $\left(4, \dfrac{1}{16}\right)$

$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \dfrac{-2(2x - 3)}{(x^2 - 3x)^3}$

$f'(4) = -\dfrac{5}{32}$

57. $f(t) = \dfrac{3t + 2}{t - 1}$, $(0, -2)$

$f'(t) = \dfrac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \dfrac{-5}{(t - 1)^2}$

$f'(0) = -5$

58. $f(x) = \dfrac{x+1}{2x-3}, \quad (2, 3)$

$$f'(x) = \frac{(2x-3)(1) - (x+1)(2)}{(2x-3)^2} = \frac{-5}{(2x-3)^2}$$

$f'(2) = -5$

59. $y = 37 - \sec^3(2x), \quad (0, 36)$

$$y' = -3\sec^2(2x)[2\sec(2x)\tan(2x)]$$
$$= -6\sec^3(2x)\tan(2x)$$

$y'(0) = 0$

60. $y = \dfrac{1}{x} + \sqrt{\cos x}, \quad \left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$y'(\pi/2)$ is undefined.

61. (a) $f(x) = \sqrt{3x^2 - 2}, \quad (3, 5)$

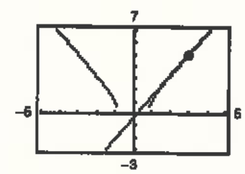
$$f'(x) = \frac{1}{2}(3x^2 - 2)^{-1/2}(6x)$$
$$= \frac{3x}{\sqrt{3x^2-2}}$$

$f'(3) = \dfrac{9}{5}$

Tangent line:

$$y - 5 = \frac{9}{5}(x - 3) \Longrightarrow 9x - 5y - 2 = 0$$

(b)



62. (a) $f(x) = \dfrac{1}{3}x\sqrt{x^2+5}, \quad (2, 2)$

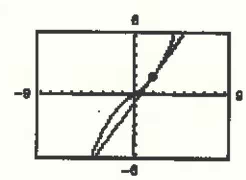
$$f'(x) = \frac{1}{3}x\left[\frac{1}{2}(x^2+5)^{-1/2}(2x)\right] + \frac{1}{3}(x^2+5)^{1/2}$$
$$= \frac{x^2}{3\sqrt{x^2+5}} + \frac{1}{3}\sqrt{x^2+5}$$

$f'(2) = \dfrac{4}{3(3)} + \dfrac{1}{3}(3) = \dfrac{13}{9}$

Tangent line:

$$y - 2 = \frac{13}{9}(x - 2) \Longrightarrow 13x - 9y - 8 = 0$$

(b)



63. (a) $f(x) = \sin 2x, \quad (\pi, 0)$

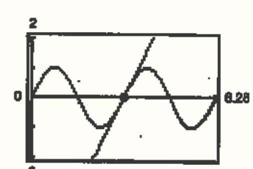
$f'(x) = 2\cos 2x$

$f'(\pi) = 2$

Tangent line:

$$y = 2(x - \pi) \Longrightarrow 2x - y - 2\pi = 0$$

(b)



64. (a) $f(x) = \tan^2 x, \left(\frac{\pi}{4}, 1\right)$

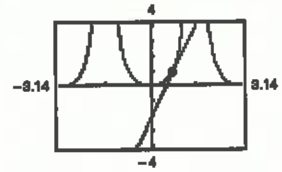
$f'(x) = 2 \tan x \sec^2 x$

$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$

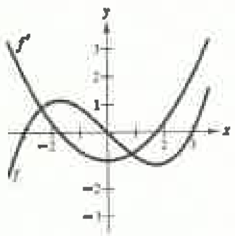
Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \Rightarrow 4x - y + (1 - \pi) = 0$$

(b)

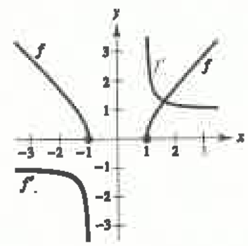


65.



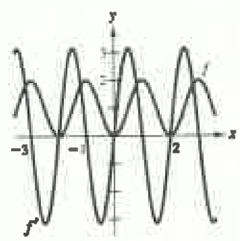
The zeros of f' correspond to the points where the graph of f has horizontal tangents.

66.



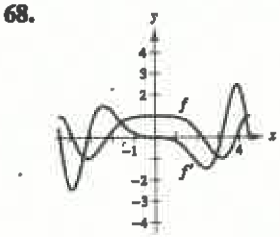
f is decreasing on $(-\infty, -1)$ so f' must be negative there. f is increasing on $(1, \infty)$ so f' must be postive there.

67.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

68.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

69. $f(x) = 2(x^2 - 1)^3$

$f'(x) = 6(x^2 - 1)^2(2x)$

$= 12x(x^4 - 2x^2 + 1)$

$= 12x^5 - 24x^3 + 12x$

$f''(x) = 60x^4 - 72x^2 + 12$

$= 12(5x^2 - 1)(x^2 - 1)$

70. $f(x) = (x - 2)^{-1}$

$f'(x) = -(x - 2)^{-2} = \dfrac{-1}{(x - 2)^2}$

$f''(x) = 2(x - 2)^{-3} = \dfrac{2}{(x - 2)^3}$

71. $f(x) = \sin x^2$

$f'(x) = 2x \cos x^2$

$f''(x) = 2x[2x(-\sin x^2)] + 2 \cos x^2 = 2[\cos x^2 - 2x^2 \sin x^2]$

72. $f(x) = \sec^2 \pi x$

$$f'(x) = 2\sec \pi x(\pi \sec \pi x \tan \pi x)$$
$$= 2\pi \sec^2 \pi x \tan \pi x$$
$$f''(x) = 2\pi \sec^2 \pi x(\sec^2 \pi x)(\pi) + 2\pi \tan \pi x(2\pi \sec^2 \pi x \tan \pi x)$$
$$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$
$$= 2\pi^2 \sec^2 \pi x(\sec^2 \pi x + 2\tan^2 \pi x)$$
$$= 2\pi^2 \sec^2 \pi x(3\sec^2 \pi x - 2)$$

73. (a) $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$
$$f'(5) = (-3)(-2) + (6)(3) = 24$$

(b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x))h'(x)$$
$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need $g'(3)$ to find $f'(5)$.

(c) $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$
$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

(d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2 g'(x)$$
$$f'(5) = 3(-3)^2(6) = 162$$

74. (a) $g(x) = \sin^2 x + \cos^2 x = 1 \Longrightarrow g'(x) = 0$

$$g'(x) = 2\sin x \cos x + 2\cos x(-\sin x) = 0$$

(b) $\tan^2 x + 1 = \sec^2 x$

$$g(x) + 1 = f(x)$$

Taking derivatives of both sides,

$$g'(x) = f'(x).$$

Equivalently, $f'(x) = 2\sec x \cdot \sec x \cdot \tan x$ and $g'(x) = 2\tan x \cdot \sec^2 x$, which are the same.

75. (a) $f = 132{,}400(331 - v)^{-1}$

$$f' = (-1)(132{,}400)(331 - v)^{-2}(-1)$$
$$= \frac{132{,}400}{(331 - v)^2}$$

When $v = 30$, $f' \approx 1.461$.

(b) $f = 132{,}400(331 + v)^{-1}$

$$f' = (-1)(132{,}400)(331 + v)^{-2}(1)$$
$$= \frac{-132{,}400}{(331 + v)^2}$$

When $v = 30$, $f' \approx -1.016$.

76. $y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$

$$v = y' = \tfrac{1}{3}[-12\sin 12t] - \tfrac{1}{4}[12\cos 12t]$$
$$= -4\sin 12t - 3\cos 12t$$

When $t = \pi/8$, $y = 0.25$ feet and $v = 4$ feet per second.

77. $\theta = 0.2\cos 8t$

The maximum angular displacement is $\theta = 0.2$ (since $-1 \le \cos 8t \le 1$).

$$\frac{d\theta}{dt} = 0.2[-8\sin 8t] = -1.6\sin 8t$$

When $t = 3$, $d\theta/dt = -1.6\sin 24 \approx 1.4489$ radians per second.

78. $y = A\cos \omega t$

(a) Amplitude: $A = \dfrac{3.5}{2} = 1.75$

$$y = 1.75\cos \omega t$$

Period: $10 \Longrightarrow \omega = \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

$$y = 1.75\cos\frac{\pi t}{5}$$

(b) $v = y' = 1.75\left[-\dfrac{\pi}{5}\sin\dfrac{\pi t}{5}\right]$

$$= -0.35\,\pi\sin\frac{\pi t}{5}$$

79. $S = C(R^2 - r^2)$

$$\frac{dS}{dt} = C\left(2R\frac{dR}{dt} - 2r\frac{dr}{dt}\right)$$

Since r is constant, we have $dr/dt = 0$ and

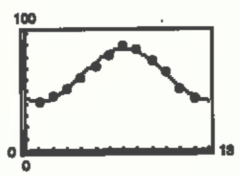
$$\frac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$$

$$= 4.224 \times 10^{-2} = 0.04224.$$

80. (a) Using a graphing utility, or by trial and error, you obtain a model of the form

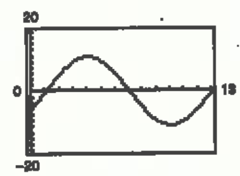
$$T(t) = 64.18 - 22.15\sin\left(\frac{\pi t}{6} + 1\right)$$

(b)



(c) $T'(t) = -22.15\cos\left(\dfrac{\pi t}{6} + 1\right)\left(\dfrac{\pi}{6}\right)$

$$= -11.60\cos\left(\frac{\pi t}{6} + 1\right)$$



(d) The temperature changes most rapidly when $t \approx 4.1$ (April) and $t \approx 10.1$ (October). The temperature changes most slowly ($T'(t) = 0$) when $t \approx 1.1$ (January) and $t \approx 7.1$ (July).

81. (a) $g(x) = f(x) - 2 \Longrightarrow g'(x) = f'(x)$

(b) $h(x) = 2f(x) \Longrightarrow h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \Longrightarrow r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know $f'(-3x)$.

$$r'(0) = -3f'(0) = (-3)\left(-\tfrac{1}{3}\right) = 1$$

$$r'(-1) = -3f'(3) = (-3)(-4) = 12$$

(d) $s(x) = f(x + 2) \Longrightarrow s'(x) = f'(x + 2)$

Hence, you need to know $f'(x + 2)$.

$$s'(-2) = f'(0) = -\tfrac{1}{3}, \text{ etc.}$$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

82. $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$f''(x) = -\beta^2 \sin \beta x$

$f'''(x) = -\beta^3 \cos \beta x$

$f^{(4)} = \beta^4 \sin \beta x$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$

83. $f(x + p) = f(x)$ for all x.

(a) Yes, $f'(x + p) = f'(x)$, which shows that f' is periodic as well.

(b) Yes, let $g(x) = f(2x)$, so $g'(x) = 2f'(2x)$. Since f' is periodic, so is g'.

84. If $f(-x) = -f(x)$, then

$$\frac{d}{dx}[f(-x)] = \frac{d}{x}[-f(x)]$$

$$f'(-x)(-1) = -f'(x)$$

$$f'(-x) = f'(x).$$

Thus, $f'(x)$ is even.

85. $g = \sqrt{x(x + n)}$

$$= \sqrt{x^2 + nx}$$

$$\frac{dg}{dx} = \frac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$$

$$= \frac{2x + n}{2\sqrt{x^2 + nx}}$$

$$= \frac{(2x + n)/2}{\sqrt{x(x + n)}}$$

$$= \frac{[x + (x + n)]/2}{\sqrt{x(x + n)}}$$

$$= \frac{a}{g}$$

86. $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}\left[\sqrt{u^2}\right] = \frac{1}{2}(u^2)^{-1/2}(2uu') = \frac{uu'}{\sqrt{u^2}} = u'\frac{u}{|u|}, u \neq 0$$

87. $g(x) = |2x - 3|$

$$g'(x) = 2\left(\frac{2x - 3}{|2x - 3|}\right), \quad x \neq \frac{3}{2}$$

88. $f(x) = |x^2 - 4|$

$$f'(x) = 2x\left(\frac{x^2 - 4}{|x^2 - 4|}\right), x \neq \pm 2$$

89. $h(x) = |x|\cos x$

$$h'(x) = -|x| \sin x + \frac{x}{|x|}\cos x, \quad x \neq 0$$

90. $f(x) = |\sin x|$

$$f'(x) = \cos x\left(\frac{\sin x}{|\sin x|}\right), x \neq k\pi$$

91. (a) $f(x) = \sin\frac{x}{2}$ $\qquad f(\pi) = \sin\frac{\pi}{2} = 1$

$f'(x) = \frac{1}{2}\cos\frac{x}{2}$ $\qquad f'(\pi) = \frac{1}{2}\cos\frac{\pi}{2} = 0$

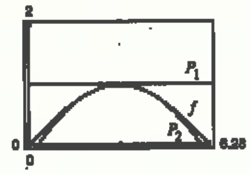
$f''(x) = -\frac{1}{4}\sin\frac{x}{2}$ $\qquad f''(\pi) = -\frac{1}{4}\sin\frac{\pi}{2} = -\frac{1}{4}$

$P_1(x) = f'(\pi)(x - \pi) + f(\pi) = f(\pi) = 1$

$$P_2(x) = \frac{1}{2}f''(\pi)(x - \pi)^2 + f'(\pi)(x - \pi) + f(\pi)$$

$$= \frac{1}{2}\left(-\frac{1}{4}\right)(x - \pi)^2 + 1 = 1 - \frac{1}{8}(x - \pi)^2$$

(b)



(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move away from $x = \pi$.

92. (a) $f(x) = \sec(2x)$

$f'(x) = 2(\sec 2x)(\tan 2x)$

$f''(x) = 2[2(\sec 2x)(\tan 2x)]\tan 2x + 2(\sec 2x)(\sec^2 2x)(2)$

$= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x]$

$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$

$f'\left(\frac{\pi}{6}\right) = 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$

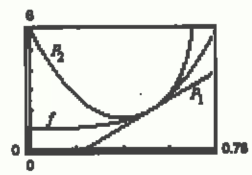
$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$

$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$

$$P_2(x) = \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

(b)



(c) P_2 is a better approximation than P_1.

(d) The accuracy worsens as you move away from $x = \pi/6$.

93. False. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$.

94. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2\cos 2x)$.

95. True

Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 16$

$2x + 2yy' = 0$

$y' = \dfrac{-x}{y}$

2. $x^2 - y^2 = 16$

$2x - 2yy' = 0$

$y' = \dfrac{x}{y}$

3. $x^{1/2} + y^{1/2} = 9$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

$y' = -\dfrac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\dfrac{y}{x}}$

4. $x^3 + y^3 = 8$

$3x^2 + 3y^2y' = 0$

$y' = -\dfrac{x^2}{y^2}$

5. $x^3 - xy + y^2 = 4$

$3x^2 - xy' - y + 2yy' = 0$

$(2y - x)y' = y - 3x^2$

$y' = \dfrac{y - 3x^2}{2y - x}$

6. $x^2y + y^2x = -2$

$x^2y' + 2xy + y^2 + 2yxy' = 0$

$(x^2 + 2xy)y' = -(y^2 + 2xy)$

$y' = \dfrac{-y(y + 2x)}{x(x + 2y)}$

7. $x^3y^3 - y - x = 0$

$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$y' = \dfrac{1 - 3x^2y^3}{3x^3y^2 - 1}$

8. $(xy)^{1/2} - x + 2y = 0$

$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$

$\dfrac{x}{2\sqrt{xy}}y' + \dfrac{y}{2\sqrt{xy}} - 1 + 2y' = 0$

$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}\,y' = 0$

$y' = \dfrac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$

9. $x^3 - 2x^2y + 3xy^2 = 38$

$3x^2 - 2x^2y' - 4xy + 6xyy' + 3y^2 = 0$

$2x(3y - x)y' = 4xy - 3x^2 - 3y^2$

$y' = \dfrac{4xy - 3x^2 - 3y^2}{2x(3y - x)}$

10. $2 \sin x \cos y = 1$

$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$

$y' = \dfrac{\cos x \cos y}{\sin x \sin y}$

$= \cot x \cot y$

11. $\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$y' = \dfrac{\cos x}{4 \sin 2y}$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$

$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$

$y' = \dfrac{\cos \pi x}{\sin \pi y}$

13. $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

14. $\cot y = x - y$

$$(-\csc^2 y)y' = 1 - y'$$

$$y' = \frac{1}{1 - \csc^2 y}$$

$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

15. $y = \sin(xy)$

$$y' = [xy' + y]\cos(xy)$$

$$y' - x\cos(xy)y' = y\cos(xy)$$

$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

16. $x = \sec\frac{1}{y}$

$$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{y}$$

$$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$$

17. $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

At $(-4, -1)$: $y' = -\frac{1}{4}$

18. $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At $(1, 1)$: $y' = \frac{2}{3}$.

19. $y^2 = \frac{x^2 - 9}{x^2 + 9}$

$$2yy' = \frac{(x^2 + 9)(2x) - (x^2 - 9)2x}{(x^2 + 9)^2}$$

$$y' = \frac{18x}{(x^2 + 9)^2 y}$$

At $(3, 0)$: y' is undefined.

20. $(x + y)^3 = x^3 + y^3$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At $(-1, 1)$: $y' = -1$.

21. $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At $(8, 1)$: $y' = -\frac{1}{2}$.

22. $x^3 + y^3 - 2xy = 0$

$$3x^2 + 3y^2y' - 2xy' - 2y = 0$$

$$(3y^2 - 2x)y' = 2y - 3x^2$$

$$y' = \frac{2y - 3x^2}{3y^2 - 2x}$$

At $(1, 1)$: $y' = -1$.

23. $\tan(x + y) = x$

$$(1 + y')\sec^2(x + y) = 1$$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1}$$

$$= -\frac{x^2}{x^2 + 1}$$

At $(0, 0)$: $y' = 0$.

24. $x\cos y = 1$

$$x[-y'\sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x\sin y}$$

$$= \frac{1}{x}\cot y = \frac{\cot y}{x}$$

At $\left(2, \frac{\pi}{3}\right)$: $y' = \frac{1}{2\sqrt{3}}$.

25. $\sqrt{x} + \sqrt{y} = 3$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$y' = -\frac{(1/2)x^{-1/2}}{(1/2)y^{-1/2}} = -\frac{\sqrt{y}}{\sqrt{x}}$$

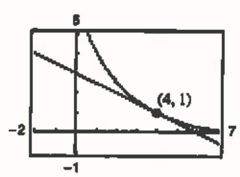
At $(4, 1)$: $y' = -\frac{1}{2}$.

Tangent line:

$$y - 1 = -\frac{1}{2}(x - 4)$$

$$y = -\frac{1}{2}x + 3$$

$$x + 2y - 6 = 0$$



26. $y^2 = \frac{x - 1}{x^2 + 1}$

$$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$$

$$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$$

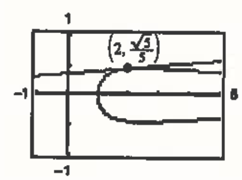
At $\left(2, \frac{\sqrt{5}}{5}\right)$: $y' = \frac{1 + 4 - 4}{[(2\sqrt{5})/5](4 + 1)^2} = \frac{1}{10\sqrt{5}}$.

Tangent line:

$$y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$$

$$10\sqrt{5}y - 10 = x - 2$$

$$x - 10\sqrt{5}y + 8 = 0$$



27. $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At $(2, 1)$: $y' = \frac{-32}{64} = -\frac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \frac{8}{x^2 + 4}\right)$

28. $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At $(2, 2)$: $y' = 2$.

29.
$$(x^2 + y^2)^2 = 4x^2y$$
$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$
$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$
$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$
$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$
$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

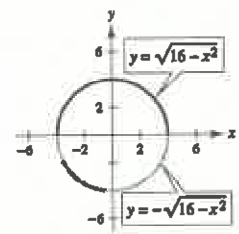
At $(1, 1)$: $y' = 0$.

30.
$$x^3 + y^3 - 6xy = 0$$
$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$
$$y'(3y^2 - 6x) = 6y - 3x^2$$
$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $\left(\frac{4}{3}, \frac{8}{3}\right)$: $y' = \frac{(16/3) - (16/9)}{(64/9) - (8/3)} = \frac{32}{40} = \frac{4}{5}$.

31. (a) $x^2 + y^2 = 16$
$$y^2 = 16 - x^2$$
$$y = \pm\sqrt{16 - x^2}$$

(b)

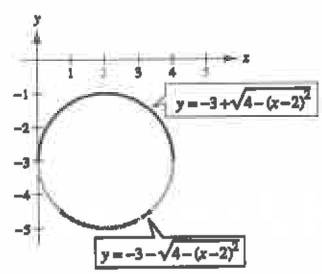


(c) Explicitly:
$$\frac{dy}{dx} = \pm\frac{1}{2}(16 - x^2)^{-1/2}(-2x)$$
$$= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm\sqrt{16 - x^2}} = \frac{-x}{y}$$

(d) Implicitly:
$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

32. (a) $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$
$$(x - 2)^2 + (y + 3)^2 = 4 \text{ (Circle)}$$
$$(y + 3)^2 = 4 - (x - 2)^2$$
$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)



(c) Explicitly:
$$\frac{dy}{dx} = \pm\frac{1}{2}[4 - (x - 2)^2]^{-1/2}(-2)(x - 2)$$
$$= \frac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2}}$$
$$= \frac{-(x - 2)}{\pm\sqrt{4 - (x - 2)^2}}$$
$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$
$$= \frac{-(x - 2)}{y + 3}$$

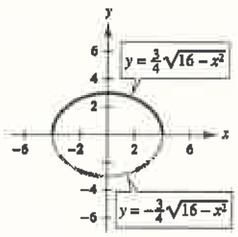
(d) Implicitly:
$$2x + 2yy' - 4 + 6y' = 0$$
$$(2y + 6)y' = -2(x - 2)$$
$$y' = \frac{-(x - 2)}{y + 3}$$

33. (a) $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm\frac{3}{4}\sqrt{16 - x^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{3}{8}(16 - x^2)^{-1/2}(-2x)$$

$$= \mp\frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y}$$

(d) Implicitly:

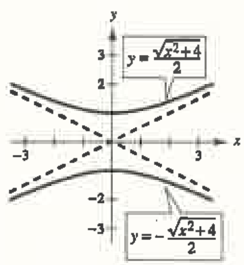
$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

34. (a) $y^2 = 1 + \dfrac{x^2}{4} = \dfrac{x^2 + 4}{4}$

$$y = \pm\frac{1}{2}\sqrt{x^2 + 4}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{4}(x^2 + 4)^{-1/2}(2x)$$

$$= \frac{\pm x}{2\sqrt{x^2 + 4}}$$

$$= \frac{\pm x}{4(1/2)\sqrt{x^2 + 4}} = \frac{x}{4y}$$

(d) Implicitly:

$$2yy' - \frac{1}{2}x = 0$$

$$y' = \frac{x}{4y}$$

35.

$$x^2 + xy = 5$$

$$2x + xy' + y = 0$$

$$y' = \frac{-(2x + y)}{x}$$

$$2 + xy'' + y' + y' = 0$$

$$xy'' = -2(1 + y')$$

$$y'' = \frac{-2[1 - (2x + y)/x]}{x} = \frac{2(x + y)}{x^2}$$

$$y'' = \frac{2(x + y)}{x^2} \cdot \frac{x}{x} = \frac{10}{x^3}$$

(**Note:** You could write $y = (5 - x^2)/x$ and calculate y' and y'' directly.)

36.
$$x^2y^2 - 2x = 3$$
$$2x^2yy' + 2xy^2 - 2 = 0$$
$$x^2yy' + xy^2 - 1 = 0$$
$$y' = \frac{1 - xy^2}{x^2y}$$
$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$
$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$
$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$$
$$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$$
$$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$$
$$y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}$$

37.
$$x^2 - y^2 = 16$$
$$2x - 2yy' = 0$$
$$y' = \frac{x}{y}$$
$$x - yy' = 0$$
$$1 - yy'' - (y')^2 = 0$$
$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$
$$y^2 - y^3y'' = x^2$$
$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

38. $1 - xy = x - y$
$$y - xy = x - 1$$
$$y = \frac{x - 1}{1 - x} = -1$$
$$y' = 0$$
$$y'' = 0$$

39. $y^2 = x^3$
$$2yy' = 3x^2$$
$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$
$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$
$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$
$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

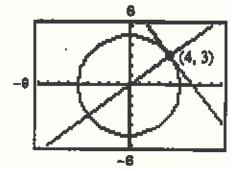
40. $y^2 = 4x$
$$2yy' = 4$$
$$y' = \frac{2}{y}$$
$$y'' = -2y^{-2}y' = \left[\frac{-2}{y^2}\right] \cdot \frac{2}{y} = \frac{-4}{y^3}$$

41. $x^2 + y^2 = 25$

$$y' = \frac{-x}{y}$$

At $(4, 3)$:

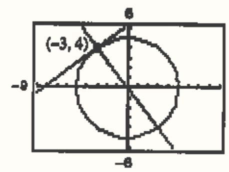
Tangent line: $y - 3 = \frac{-4}{3}(x - 4) \Longrightarrow 4x + 3y - 25 = 0$



Normal line: $y - 3 = \frac{3}{4}(x - 4) \Longrightarrow 3x - 4y = 0.$

At $(-3, 4)$:

Tangent line: $y - 4 = \frac{3}{4}(x + 3) \Longrightarrow 3x - 4y + 25 = 0$



Normal line: $y - 4 = \frac{-4}{3}(x + 3) \Longrightarrow 4x + 3y = 0.$

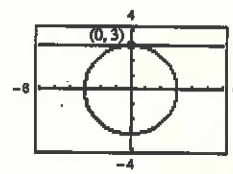
42. $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

At $(0, 3)$:

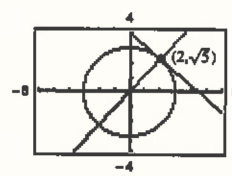
Tangent line: $y = 3$

Normal line: $x = 0.$



At $(2, \sqrt{5})$:

Tangent line: $y - \sqrt{5} = \frac{-2}{\sqrt{5}}(x - 2) \Longrightarrow 2x + \sqrt{5}y - 9 = 0$



Normal line: $y - \sqrt{5} = \frac{\sqrt{5}}{2}(x - 2) \Longrightarrow \sqrt{5}x - 2y = 0.$

43. $x^2 + y^2 = r^2$

$2x + 2yy' = 0$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

44. $y^2 = 4x$

$2yy' = 4$

$y' = \dfrac{2}{y} = 1$ at $(1, 2)$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x-1)^2 + [(3-x)-2]^2 = 16$$
$$2(x-1)^2 = 16$$
$$x = 1 \pm 2\sqrt{2}.$$

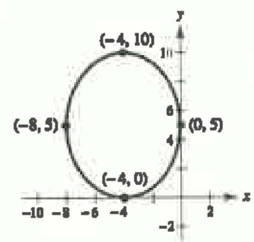
Centers of the circles: $\left(1 + 2\sqrt{2}, 2 - 2\sqrt{2}\right)$ and $\left(1 - 2\sqrt{2}, 2 + 2\sqrt{2}\right)$

Equations: $\left(x - 1 - 2\sqrt{2}\right)^2 + \left(y - 2 + 2\sqrt{2}\right)^2 = 16$

$\left(x - 1 + 2\sqrt{2}\right)^2 + \left(y - 2 - 2\sqrt{2}\right)^2 = 16$

45. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$
$$y' = \frac{200 + 50x}{160 - 32y}$$



Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$
$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

Horizontal tangents: $(-4, 0), (-4, 10)$.

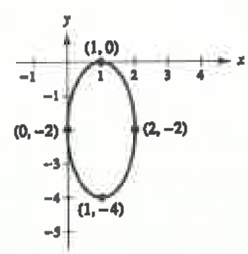
Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$
$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

Vertical tangents: $(0, 5), (-8, 5)$.

46. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$$8x + 2yy' - 8 + 4y' = 0$$
$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$



Horizontal tangents occur when $x = 1$:

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$
$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

Horizontal tangents: $(1, 0), (1, -4)$.

Vertical tangents occur when $y = -2$:

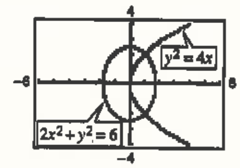
$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$
$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2), (2, -2)$.

47. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.



Ellipse:	Parabola:
$4x + 2yy' = 0$	$2yy' = 4$
$y' = -\dfrac{2x}{y}$	$y' = \dfrac{2}{y}$

At (1, 2), the slopes are:

$y' = -1$ $\qquad$ $y' = 1.$

At $(1, -2)$, the slopes are:

$y' = 1$ $\qquad$ $y' = -1.$

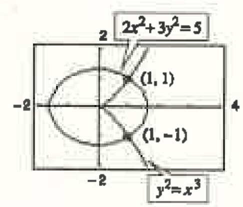
Tangents are perpendicular.

48. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$2x^2 + 3x^3 = 5$ and $3x^3 + 2x^2 - 5 = 0$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$



$y^2 = x^3$:	$2x^2 + 3y^2 = 5$:
$2yy' = 3x^2$	$4x + 6yy' = 0$
$y' = \dfrac{3x^2}{2y}$	$y' = -\dfrac{2x}{3y}$

At (1, 1), the slopes are:

$y' = \dfrac{3}{2}$ $\qquad$ $y' = -\dfrac{2}{3}.$

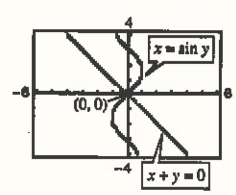
At $(1, -1)$, the slopes are:

$y' = -\dfrac{3}{2}$ $\qquad$ $y' = \dfrac{2}{3}.$

Tangents are perpendicular.

49. $y = -x$ and $x = \sin y$

Point of intersection: (0, 0)



$y = -x$:	$x = \sin y$:
$y' = -1$	$1 = y' \cos y$
	$y' = \sec y$

At (0, 0), the slopes are:

$y' = -1$ $\qquad$ $y' = 1.$

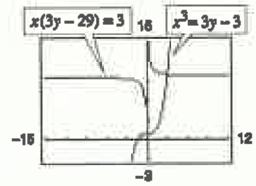
Tangents are perpendicular.

50. Rewriting each equation and differentiating,

$$x^3 = 3(y-1) \qquad x(3y-29) = 3$$

$$y = \frac{x^3}{3} + 1 \qquad y = \frac{1}{3}\left(\frac{3}{x} + 29\right)$$

$$y' = x^2 \qquad y' = -\frac{1}{x^2}.$$

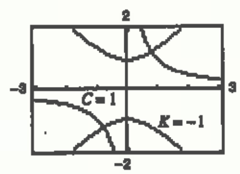


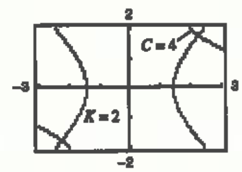
For each value of x, the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

51. $xy = C \qquad x^2 - y^2 = K$

$$xy' + y = 0 \qquad 2x - 2yy' = 0$$

$$y' = -\frac{y}{x} \qquad y' = \frac{x}{y}$$



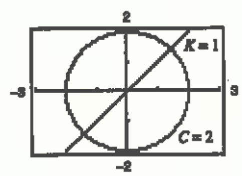


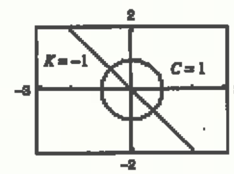
At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

52. $x^2 + y^2 = C^2 \qquad y = Kx$

$$2x + 2yy' = 0 \qquad y' = K$$

$$y' = -\frac{x}{y}$$





At the point of intersection (x, y) the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

53. $2y^2 - 3x^4 = 0$

(a) $4yy' - 12x^3 = 0$

$$4yy' = 12x^3$$

$$y' = \frac{12x^3}{4y} = \frac{3x^3}{y}$$

(b) $4y\dfrac{dy}{dt} - 12x^3\dfrac{dx}{dt} = 0$

$$y\frac{dy}{dt} = 3x^3\frac{dx}{dt}$$

54. $x^2 - 3xy^2 + y^3 = 10$

(a) $2x - 3y^2 - 6xyy' + 3y^2y' = 0$

$$(-6xy + 3y^2)y' = 3y^2 - 2x$$

$$y' = \frac{3y^2 - 2x}{3y^2 - 6xy}$$

(b) $2x\dfrac{dx}{dt} - 3y^2\dfrac{dx}{dt} - 6xy\dfrac{dy}{dt} + 3y^2\dfrac{dy}{dt} = 0$

$$(2x - 3y^2)\frac{dx}{dt} = (6xy - 3y^2)\frac{dy}{dt}$$

55. $\cos \pi y - 3 \sin \pi x = 1$

(a) $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

$$y' = \frac{-3 \cos \pi x}{\sin \pi y}$$

(b) $-\pi \sin(\pi y)\frac{dy}{dt} - 3\pi \cos(\pi x)\frac{dx}{dt} = 0$

$$-\sin(\pi y)\frac{dy}{dt} = 3 \cos(\pi x)\frac{dx}{dt}$$

56. (a) $4 \sin x \cos y = 1$

$4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$

$$y' = \frac{\cos x \cos y}{\sin x \sin y}$$

(b) $4 \sin x(-\sin y)\frac{dy}{dt} + 4 \cos x\frac{dx}{dt}\cos y = 0$

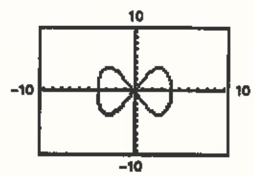
$$\cos x \cos y\frac{dx}{dt} = \sin x \sin y\frac{dy}{dt}$$

57. (a) $x^4 = 4(4x^2 - y^2)$

$4y^2 = 16x^2 - x^4$

$y^2 = 4x^2 - \frac{1}{4}x^4$

$y = \pm\sqrt{4x^2 - \frac{1}{4}x^4}$



(b) $y = 3 \Longrightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$36 = 16x^2 - x^4$

$x^4 - 16x^2 + 36 = 0$

$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = (1 \pm \sqrt{7})^2$.

Hence, there are four values of x:

$-1-\sqrt{7}, 1-\sqrt{7}, -1+\sqrt{7}, 1 + \sqrt{7}$

To find the slope, $2yy' = 8x - x^3 \Longrightarrow y' = \frac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} + 7)$, and the line is

$y_1 = \frac{1}{3}(\sqrt{7} + 7)(x + 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} + 7)x + 8\sqrt{7} + 23]$.

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} - 7)$, and the line is

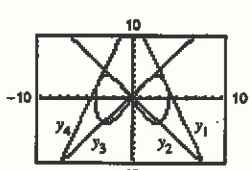
$y_2 = \frac{1}{3}(\sqrt{7} - 7)(x - 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} - 7)x + 23 - 8\sqrt{7}]$.

For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} - 7)$, and the line is

$y_3 = -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} - 7)x - (23 - 8\sqrt{7})]$.

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} + 7)$, and the line is

$y_4 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} + 7)x - (8\sqrt{7} + 23)]$.



—CONTINUED—

57. —CONTINUED—

(c) Equating y_3 and y_4,

$$-\frac{1}{3}(\sqrt{7}-7)(x+1-\sqrt{7})+3=-\frac{1}{3}(\sqrt{7}+7)(x-1-\sqrt{7})+3$$

$$(\sqrt{7}-7)(x+1-\sqrt{7})=(\sqrt{7}+7)(x-1-\sqrt{7})$$

$$\sqrt{7}x+\sqrt{7}-7-7x-7+7\sqrt{7}=\sqrt{7}x-\sqrt{7}-7+7x-7-7\sqrt{7}$$

$$16\sqrt{7}=14x$$

$$x=\frac{8\sqrt{7}}{7}$$

If $x=\frac{8\sqrt{7}}{7}$, then $y=5$ and the lines intersect at $\left(\frac{8\sqrt{7}}{7}, 5\right)$.

58. $\tan y = x$

$$y'\sec^2 y = 1$$

$$y'=\frac{1}{\sec^2 y},\ -\frac{\pi}{2}<y<\frac{\pi}{2}$$

$$\sec^2 y = 1+\tan^2 y = 1+x^2$$

$$y'=\frac{1}{1+x^2}$$

59. Let $f(x)=x^n=x^{p/q}$, where p and q are nonzero integers and $q>0$. First consider the case where $p=1$. The derivative of $f(x)=x^{1/q}$ is given by

$$\frac{d}{dx}[x^{1/q}]=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

where $t=x+\Delta x$. Observe that

$$\frac{f(t)-f(x)}{t-x}=\frac{t^{1/q}-x^{1/q}}{t-x}=\frac{t^{1/q}-x^{1/q}}{(t^{1/q})^q-(x^{1/q})^q}$$

$$=\frac{t^{1/q}-x^{1/q}}{(t^{1/q}-x^{1/q})(t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdots+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)})}$$

$$=\frac{1}{t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdots+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)}}.$$

As $t\to x$, the denominator approaches $qx^{1-(1/q)}$. That is,

$$\frac{d}{dx}[x^{1/q}]=\frac{1}{qx^{1-(1/q)}}=\frac{1}{q}x^{(1/q)-1}.$$

Now consider $f(x)=x^{p/q}=(x^p)^{1/q}$. From the Chain Rule,

$$f'(x)=\frac{1}{q}(x^p)^{(1/q)-1}\frac{d}{dx}[x^p]$$

$$=\frac{1}{q}(x^p)^{(1/q)-1}px^{p-1}=\frac{p}{q}x^{[(p/q)-p]+(p-1)}=\frac{p}{q}x^{(p/q)-1}=nx^{n-1}\ \left(n=\frac{p}{q}\right).$$

60. $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0):

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

x-intercept: $\left(x_0 + \sqrt{x_0}\sqrt{y_0}, 0\right)$

y-intercept: $\left(0, y_0 + \sqrt{x_0}\sqrt{y_0}\right)$

Sum of intercepts:

$$\left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

Section 2.6 Related Rates

1. $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x}\frac{dy}{dt}$$

(a) When $x = 4$ and $dx/dt = 3$,

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}.$$

(b) When $x = 25$ and $dy/dt = 2$,

$$\frac{dx}{dt} = 2\sqrt{25}\,(2) = 20.$$

2. $y = x^2 - 3x$

$$\frac{dy}{dt} = (2x - 3)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(\frac{1}{2x - 3}\right)\frac{dy}{dt}$$

(a) When $x = 3$ and $dx/dt = 2$,

$$\frac{dy}{dt} = [2(3) - 3](2) = 6.$$

(b) When $x = 1$ and $dy/dt = 5$,

$$\frac{dx}{dt} = \frac{1}{2(1) - 3}(5) = -5.$$

3. $xy = 4$

$$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right)\frac{dy}{dt}$$

(a) When $x = 8$, $y = 1/2$, and $dx/dt = 10$,

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}.$$

(b) When $x = 1$, $y = 4$, and $dy/dt = -6$,

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}.$$

4. $x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right)\frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}.$$

5. $y = x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x\,\frac{dx}{dt}$$

(a) When $x = -1$,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When $x = 1$,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

(d) When $x = 3$,

$$\frac{dy}{dt} = 2(3)(2) = 12 \text{ cm/sec.}$$

6. $y = \dfrac{1}{1 + x^2}$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \left[\frac{-2x}{(1 + x^2)^2}\right]\frac{dx}{dt}$$

(a) When $x = -2$,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25} \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 0 \text{ cm/sec.}$$

(c) When $x = 2$,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = \frac{-8}{25} \text{ cm/sec.}$$

(d) When $x = 10$,

$$\frac{dy}{dt} = \frac{-2(10)(2)}{(101)^2} = \frac{-40}{10{,}201} \approx -0.0039 \text{ cm/sec.}$$

7. $y = \tan x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \sec^2 x\,\frac{dx}{dt}$$

(a) When $x = -\pi/3$,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When $x = -\pi/4$,

$$\frac{dy}{dt} = (\sqrt{2})^2(2) = 4 \text{ cm/sec.}$$

(c) When $x = 0$,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

(d) When $x = 1$,

$$\frac{dy}{dt} = (\sec 1)^2 2 \approx 6.8510 \text{ cm/sec.}$$

8. $y = \sin x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \cos x\frac{dx}{dt}$$

(a) When $x = \pi/6$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{6}\right)(2) = \sqrt{3} \text{ cm/sec.}$$

(b) When $x = \pi/4$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{4}\right)(2) = \sqrt{2} \text{ cm/sec.}$$

(c) When $x = \pi/3$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{3}\right)(2) = 1 \text{ cm/sec.}$$

(d) When $x = \pi/2$,

$$\frac{dy}{dt} = \left(\cos\frac{\pi}{2}\right)(2) = 0 \text{ cm/sec.}$$

9. (a) For increasing x, dy/dt decreases.

(b) For increasing y, dx/dt increases.

10. (a) For increasing x, dy/dt increases.

(b) For increasing y, dx/dt decreases.

11. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$\frac{dx}{dt} = 2$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x)\frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}}\frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

12. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$

$\frac{dx}{dt} = 2$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2\sin x \cos x)\frac{dx}{dt} = \frac{x + \sin x \cos x}{\sqrt{x^2 + \sin^2 x}}\frac{dx}{dt} = \frac{2 + 2\sin x \cos x}{\sqrt{x^2 + \sin^2 x}}$$

13. $A = \pi r^2$

$\frac{dr}{dt} = 2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

(a) When $r = 6$,

$\frac{dA}{dt} = 2\pi(6)(2) = 24\pi \text{ in}^2/\text{min}.$

(b) When $r = 24$,

$\frac{dA}{dt} = 2\pi(24)(2) = 96\pi \text{ in}^2/\text{min}.$

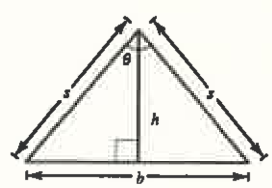
14. $A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

If dr/dt is constant, dA/dt is not constant.

$\frac{dA}{dt}$ depends on r.

15. (a) $\sin\frac{\theta}{2} = \frac{(1/2)b}{s} \Longrightarrow b = 2s\sin\frac{\theta}{2}$



$\cos\frac{\theta}{2} = \frac{h}{s} \Longrightarrow h = s\cos\frac{\theta}{2}$

$$A = \frac{1}{2}bh = \frac{1}{2}\left(2s\sin\frac{\theta}{2}\right)\left(s\cos\frac{\theta}{2}\right)$$

$$= \frac{s^2}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{s^2}{2}\sin\theta$$

(b) $\frac{dA}{dt} = \frac{s^2}{2}\cos\theta\frac{d\theta}{dt}$ where $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min.

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$ rad/min.

When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{s^2}{8}$ rad/min.

(c) If $d\theta/dt$ is constant, dA/dt is proportional to $\cos\theta$.

16. $V = \frac{4}{3}\pi r^3$

$\frac{dr}{dt} = 2$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

(a) When $r = 6$, $\frac{dV}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{min}$.

When $r = 24$, $\frac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi \text{ in}^3/\text{min}$.

(b) If dr/dt is constant, dV/dt is proportional to r^2.

17. $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = 500$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right) = \frac{1}{4\pi r^2}(500)$

(a) When $r = 30$, $\frac{dr}{dt} = \frac{1}{4\pi(30)^2}(500) = \frac{5}{36\pi}$ cm/min.

(b) When $r = 60$, $\frac{dr}{dt} = \frac{1}{4\pi(60)^2}(500) = \frac{5}{144\pi}$ cm/min.

18. $V = x^3$

$\frac{dx}{dt} = 3$

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

(a) When $x = 1$,

$\frac{dV}{dt} = 3(1)^2(3) = 9 \text{ cm}^3/\text{sec}$.

(b) When $x = 10$,

$\frac{dV}{dt} = 3(10)^2(3) = 900 \text{ cm}^3/\text{sec}$.

19. $s = 6x^2$

$\frac{dx}{dt} = 3$

$\frac{ds}{dt} = 12x \frac{dx}{dt}$

(a) When $x = 1$,

$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec}$.

(b) When $x = 10$,

$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec}$.

20. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$

$\frac{dr}{dt} = 2$

$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$

(a) When $r = 6$,

$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min}$.

(b) When $r = 24$,

$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min}$.

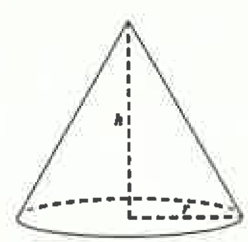
21. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{4}h^2\right)h \quad [\text{since } 2r = 3h]$

$= \frac{3\pi}{4}h^3$

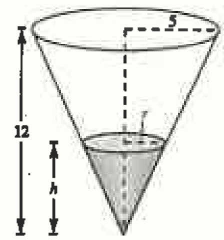
$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$

When $h = 15$, $\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.



22. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144}h^3 = \frac{25\pi}{3(144)}h^3$



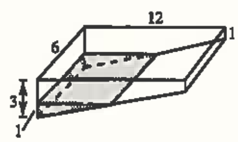
$\left(\text{By similar triangles, } \frac{r}{5} = \frac{h}{12} \Rightarrow r = \frac{5}{12}h.\right)$

$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{25\pi}{144}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right)\frac{dV}{dt}$

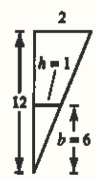
When $h = 8$, $\frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi}$ ft/min.

23.



(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1m. of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$



(see similar triangle diagram)

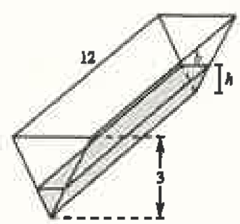
% pool filled $= \frac{18}{144}(100\%) = 12.5\%$

(b) Since for $0 \le h \le 2$, $b = 6h$, you have

$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$

$\frac{dV}{dt} = 36h\frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144}$ m/min.

24. $V = \left(\frac{1}{2}\right)bh(12) = 6bh = 6h^2$ (since $b = h$)

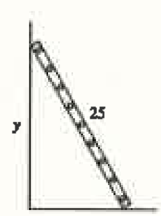


$\frac{dV}{dt} = 12h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \left(\frac{1}{12h}\right)\frac{dV}{dt}$

When $h = 1$ and $dV/dt = 2$,

$\frac{dh}{dt} = \frac{1}{12}(2) = \frac{1}{6}$ ft/min $= 2$ in/min.

25. $x^2 + y^2 = 25^2$



$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y}$ since $\frac{dx}{dt} = 2$.

(a) When $x = 7$, $y = \sqrt{576} = 24$, $\frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12}$ ft/sec.

When $x = 15$, $y = \sqrt{400} = 20$, $\frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2}$ ft/sec.

When $x = 24$, $y = 7$, $\frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7}$ ft/sec.

—CONTINUED—

25. —CONTINUED—

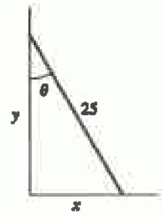
(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

From part (a) we have $x = 7, y = 24, \frac{dx}{dt} = 2$, and $\frac{dy}{dt} = -\frac{7}{12}$.

Thus, $\frac{dA}{dt} = \frac{1}{2}\left[7\left(-\frac{7}{12}\right) + 24(2)\right] = \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec}$.

(c) $\tan\theta = \frac{x}{y}$



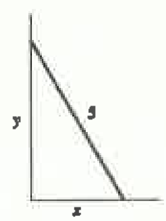
$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left[\frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}\right]$$

Using $x = 7, y = 24, \frac{dx}{dt} = 2, \frac{dy}{dt} = -\frac{7}{12}$ and $\cos\theta = \frac{24}{25}$, we have

$$\frac{d\theta}{dt} = \left(\frac{24}{25}\right)^2\left[\frac{1}{24}(2) - \frac{7}{(24)^2}\left(-\frac{7}{12}\right)\right] = \frac{1}{12}\text{ rad/sec.}$$

26. $x^2 + y^2 = 25$



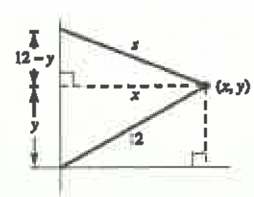
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x}\cdot\frac{dy}{dt} = -\frac{0.15y}{x}\text{ since }\frac{dy}{dt} = 0.15.$$

When $x = 8$,

$$y = \sqrt{18.75}, \frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5}0.15 \approx -0.26\text{ m/sec}$$

27. When $y = 6, x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and



$$s = \sqrt{x^2 + (12 - y)^2}$$

$$= \sqrt{108 + 36} = 12.$$

$$x^2 + (12 - y)^2 = s^2$$

$$2x\frac{dx}{dt} + 2(12 - y)(-1)\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} + (y - 12)\frac{dy}{dt} = s\frac{ds}{dt}$$

Also, $x^2 + y^2 = 12^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $x\frac{dx}{dt} + (y - 12)\left(\frac{-x}{y}\frac{dx}{dt}\right) = s\frac{ds}{dt}$

$$\frac{dx}{dt}\left[x - x + \frac{12x}{y}\right] = s\frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x}\cdot\frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})}(-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15}\text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt} = \frac{-6\sqrt{3}}{6}\cdot\frac{(-\sqrt{3})}{15} = \frac{1}{5}\text{ m/sec (vertical).}$$

28. Let L be the length of the rope.

$$L^2 = 144 + x^2$$

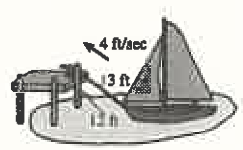
$$2L\frac{dL}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{L}{x}\cdot\frac{dL}{dt} = -\frac{4L}{x} \text{ since } \frac{dL}{dt} = -4 \text{ ft/sec.}$$

When $L = 13$,

$$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$$

$$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4 \text{ ft/sec.}$$



Speed of the boat increases as it approaches the dock.

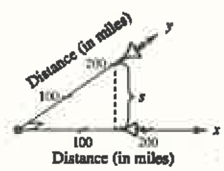
29. (a) $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$



When $x = 150$ and $y = 200$, $s = 250$ and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

(b) $t = \dfrac{250}{750} = \dfrac{1}{3}\text{ hr} = 20 \text{ min}$

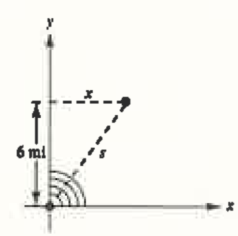
30. $x^2 + y^2 = s^2$

$$x = \sqrt{s^2 - y^2}$$

$$\frac{ds}{dt} = -240 \text{ mph}$$

$$y = 6 \text{ mi}$$

$$x = \sqrt{s^2 - 36}$$



$$\frac{dx}{dt} = \left(\frac{1}{2}\right)(s^2 - 36)^{-1/2}\left(2s\frac{ds}{dt}\right) = \frac{s}{\sqrt{s^2 - 36}}\cdot\frac{ds}{dt}$$

When $s = 10$, $\dfrac{dx}{dt} = \dfrac{10}{\sqrt{10^2 - 36}}(-240)$

$$= \frac{10}{8}(-240) = -300 \text{ mph.}$$

The speed of the plane is 300 mph.

31. $s^2 = 90^2 + x^2$

$x = 30$

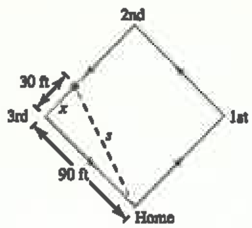
$\dfrac{dx}{dt} = -28$

$2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt} \Longrightarrow \dfrac{ds}{dt} = \dfrac{x}{s} \cdot \dfrac{dx}{dt}$

When $x = 30$,

$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$

$\dfrac{ds}{dt} = \dfrac{30}{30\sqrt{10}}(-28) = \dfrac{-28}{\sqrt{10}} \approx -8.85$ ft/sec.



32. $s^2 = 90^2 + x^2$

$x = 60$

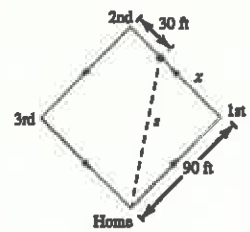
$\dfrac{dx}{dt} = 28$

$\dfrac{ds}{dt} = \dfrac{x}{s} \cdot \dfrac{dx}{dt}$

When $x = 60$,

$s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$

$\dfrac{ds}{dt} = \dfrac{60}{30\sqrt{13}}(28) = \dfrac{56}{\sqrt{13}} \approx 15.53$ ft/sec.



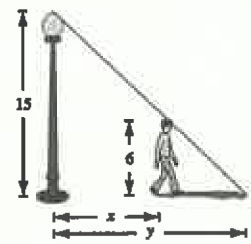
33. (a) $\dfrac{15}{6} = \dfrac{y}{y - x} \Longrightarrow 15y - 15x = 6y$

$y = \dfrac{5}{3}x$

$\dfrac{dx}{dt} = 5$

$\dfrac{dy}{dt} = \dfrac{5}{3} \cdot \dfrac{dx}{dt} = \dfrac{5}{3}(5) = \dfrac{25}{3}$ ft/sec

(b) $\dfrac{d(y - x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt} = \dfrac{25}{3} - 5 = \dfrac{10}{3}$ ft/sec



34. (a) $\dfrac{20}{6} = \dfrac{y}{y - x}$

$20y - 20x = 6y$

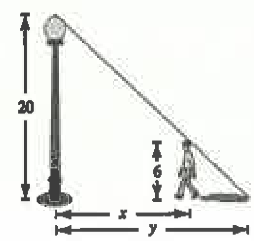
$14y = 20x$

$y = \dfrac{10}{7}x$

$\dfrac{dx}{dt} = -5$

$\dfrac{dy}{dt} = \dfrac{10}{7}\dfrac{dx}{dt} = \dfrac{10}{7}(-5) = \dfrac{-50}{7}$ ft/sec

(b) $\dfrac{d(y - x)}{dt} = \dfrac{dy}{dt} - \dfrac{dx}{dt} = \dfrac{-50}{7} - (-5) = \dfrac{-50}{7} + \dfrac{35}{7} = \dfrac{-15}{7}$ ft/sec



35. $x(t) = \frac{1}{2}\sin\frac{\pi t}{6}, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}, y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{\pi}{6}\right)\cos\frac{\pi t}{6} = \frac{\pi}{12}\cos\frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus,

$$\frac{dy}{dt} = -\frac{1/4}{\sqrt{15}/4}\cdot\frac{\pi}{12}\cos\left(\frac{\pi}{6}\right)$$

$$= \frac{-\pi}{\sqrt{15}}\left(\frac{1}{12}\right)\frac{\sqrt{3}}{2} = \frac{-\pi}{24}\frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120}$$

Speed $= \left|\frac{-\sqrt{5}\pi}{120}\right| = \frac{\sqrt{5}\pi}{120}$ m/sec

36. $x(t) = \frac{3}{5}\sin\pi t, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}, y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ m.

Lowest point: $\left(0, \frac{4}{5}\right)$

(c) When $x = \frac{3}{10}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and

$$\frac{3}{10} = \frac{3}{5}\sin\pi t \Rightarrow \sin\pi t = \frac{1}{2} \Rightarrow t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5}\pi\cos\pi t$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $\frac{dy}{dt} = \frac{-3/10}{\sqrt{15}/4}\cdot\frac{3}{5}\pi\cos\left(\frac{\pi}{6}\right)$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}.$$

Speed $= \left|\frac{-9\sqrt{5}\pi}{125}\right| \approx 0.5058$ m/sec

37. Since the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, since $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Therefore,

$$k(4\pi r^2) = 4\pi r^2\frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}.$$

38. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\frac{dR_1}{dt} = 1$$

$$\frac{dR_2}{dt} = 1.5$$

$$\frac{1}{R^2}\cdot\frac{dR}{dt} = \frac{1}{R_1^2}\cdot\frac{dR_1}{dt} + \frac{1}{R_2^2}\cdot\frac{dR_2}{dt}$$

When $R_1 = 50$ and $R_2 = 75$,

$$R = 30$$

$$\frac{dR}{dt} = (30)^2\left[\frac{1}{(50)^2}(1) + \frac{1}{(75)^2}(1.5)\right]$$

$$= 0.6 \text{ ohms/sec.}$$

39. $$pv^{1.3} = k$$

$$1.3\,pv^{0.3}\frac{dv}{dt} + v^{1.3}\frac{dp}{dt} = 0$$

$$v^{0.3}\left(1.3p\frac{dv}{dt} + v\frac{dp}{dt}\right) = 0$$

$$1.3p\frac{dv}{dt} = -v\frac{dp}{dt}$$

40. $$rg\tan\theta = v^2$$

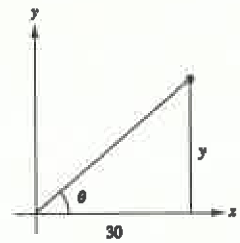
$32r\tan\theta = v^2$, r is a constant.

$$32r\sec^2\theta\frac{d\theta}{dt} = 2v\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v}\sec^2\theta\,\frac{d\theta}{dt}$$

Likewise, $\dfrac{d\theta}{dt} = \dfrac{v}{16r}\cos^2\theta\,\dfrac{dv}{dt}$.

41. $$\tan\theta = \frac{y}{30}$$



$$\frac{dy}{dt} = 3\text{ m/sec.}$$

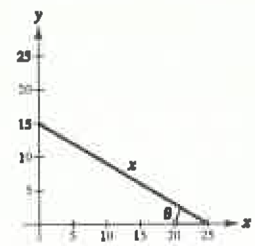
$$\sec^2\theta\cdot\frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{30}\cos^2\theta\cdot\frac{dy}{dt}$$

When $y = 30$, $\theta = \pi/4$ and $\cos\theta = \sqrt{2}/2$. Thus,

$$\frac{d\theta}{dt} = \frac{1}{30}\left(\frac{1}{2}\right)(3) = \frac{1}{20}\text{ rad/sec.}$$

42. $$\sin\theta = \frac{15}{x}$$



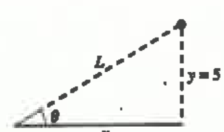
$$\frac{dx}{dt} = -1\text{ ft/sec}$$

$$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{-15}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-15}{x^2}(\sec\theta)\frac{dx}{dt}$$

$$= \frac{-15}{625}\left(\frac{25}{20}\right)(-1) = \frac{3}{100}\text{ rad/sec}$$

43. $$\tan\theta = \frac{y}{x},\ y = 5$$



$$\frac{dx}{dt} = -600\text{ mi/hr}$$

$$(\sec^2\theta)\frac{d\theta}{dt} = -\frac{5}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta\left(-\frac{5}{x^2}\right)\frac{dx}{dt} = \frac{x^2}{L^2}\left(-\frac{5}{x^2}\right)\frac{dx}{dt}$$

$$= \left(-\frac{5^2}{L^2}\right)\left(\frac{1}{5}\right)\frac{dx}{dt} = (-\sin^2\theta)\left(\frac{1}{5}\right)(-600) = 120\sin^2\theta$$

(a) When $\theta = 30°$, $\dfrac{d\theta}{dt} = \dfrac{120}{4} = 30$ rad/hr $= \dfrac{1}{2}$ rad/min.

(b) When $\theta = 60°$, $\dfrac{d\theta}{dt} = 120\left(\dfrac{3}{4}\right) = 90$ rad/hr $= \dfrac{3}{2}$ rad/min.

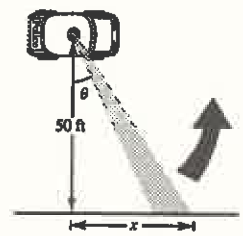
(c) When $\theta = 75°$, $\dfrac{d\theta}{dt} = 120\sin^2 75° \approx 111.96$ rad/hr ≈ 1.87 rad/min.

44. $\tan\theta = \dfrac{x}{50}$

$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = 50\sec^2\theta\left(\frac{d\theta}{dt}\right)$$

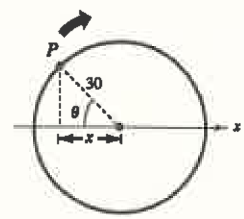


(a) When $\theta = 30°$, $\dfrac{dx}{dt} = \dfrac{200\pi}{3}$ ft/sec.

(b) When $\theta = 60°$, $\dfrac{dx}{dt} = 200\pi$ ft/sec.

(c) When $\theta = 70°$, $\dfrac{dx}{dt} \approx 427.43\pi$ ft/sec.

45. $\dfrac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi$ rad/sec

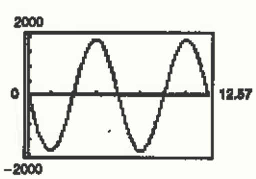


(a) $\cos\theta = \dfrac{x}{30}$

$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$$\frac{dx}{dt} = -30\sin\theta\frac{d\theta}{dt} = -30\sin\theta(20\pi) = -600\pi\sin\theta$$

(b)



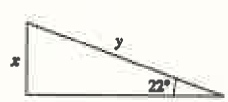
(c) $|dx/dt| = |-600\pi\sin\theta|$ is greatest when $\sin\theta = 1 \implies \theta = (\pi/2) + n\pi$ (or $90° + n \cdot 180°$).

$|dx/dt|$ is least when $\theta = n\pi$ (or $n \cdot 180°$).

(d) For $\theta = 30°$, $\dfrac{dx}{dt} = -600\pi\sin(30°) = -600\pi\dfrac{1}{2} = -300\pi$ cm/sec.

For $\theta = 60°$, $\dfrac{dx}{dt} = -600\pi\sin(60°) = -600\pi\dfrac{\sqrt{3}}{2} = -300\sqrt{3}\,\pi$ cm/sec.

46. $\sin 22° = \dfrac{x}{y}$



$$0 = -\frac{x}{y^2}\cdot\frac{dy}{dt} + \frac{1}{y}\cdot\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y}\cdot\frac{dy}{dt} = (\sin 22°)(240) \approx 89.9056 \text{ mi/hr}$$

47. $\tan\theta = \dfrac{x}{50} \implies x = 50\tan\theta$

$$\frac{dx}{dt} = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$2 = 50\sec^2\theta\,\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25}\cos^2\theta,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

48. (a) $dy/dt = 3(dx/dt)$ means that y changes three times as fast as x changes.

(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

49. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \dfrac{d^2y}{dt^2}$

First derivative: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Second derivative: $x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt} + y\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt}\cdot\dfrac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right)\left[-x\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right]$$

When $x = 7, y = 24, \dfrac{dy}{dt} = -\dfrac{7}{12}$, and $\dfrac{dx}{dt} = 2$ (see Exercise 25). Since $\dfrac{dx}{dt}$ is constant, $\dfrac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24}\left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2\right] = \frac{1}{24}\left[-4 - \frac{49}{144}\right] = \frac{1}{24}\left[-\frac{625}{144}\right] \approx -0.1808 \text{ ft/sec}^2$$

50. $L^2 = 144 + x^2$; acceleration of the boat $= \dfrac{d^2x}{dt^2}$.

First derivative: $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$

$$L\frac{dL}{dt} = x\frac{dx}{dt}$$

Second derivative: $L\dfrac{d^2L}{dt^2} + \dfrac{dL}{dt}\cdot\dfrac{dL}{dt} = x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right)\left[L\frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2\right]$$

When $L = 13, x = 5, \dfrac{dx}{dt} = -10.4$, and $\dfrac{dL}{dt} = -4$ (see Exercise 28). Since $\dfrac{dL}{dt}$ is constant, $\dfrac{d^2L}{dt^2} = 0$.

$$\frac{d^2x}{dt^2} = \frac{1}{5}[13(0) + (-4)^2 - (-10.4)^2]$$

$$= \frac{1}{5}[16 - 108.16] = \frac{1}{5}[-92.16] = -18.432 \text{ ft/sec}^2$$

51. (a) Using a graphing utility, you obtain

$$m(s) = -1.014s^2 + 31.685s - 214.437.$$

(b) $\dfrac{dm}{dt} = (-2.028s + 31.685)\dfrac{ds}{dt}$

(c) If $\dfrac{ds}{dt} = 1.2$, then $s = 16.5$ in 1995 ($t = 5$).

Then $\dfrac{dm}{dt} = (-2.028(16.5) + 31.685)(1.2) \approx -2.1.$

52. $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$

By similar triangles, $\frac{20}{x} = \frac{y}{x - 12}$

$$20x - 240 = xy.$$

When $y = 15.1$, $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

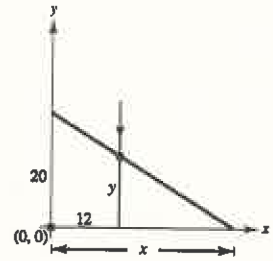
$$x = \frac{240}{4.9}.$$

$$20x - 240 = xy$$

$$20\frac{dx}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y}\frac{dy}{dt}$$

At $t = 1$, $\frac{dx}{dt} = \frac{240/4.9}{20 - 15.1}(-9.8) \approx -97.96$ m/sec.



Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2$$

2. $f(x) = \frac{x + 1}{x - 1}$

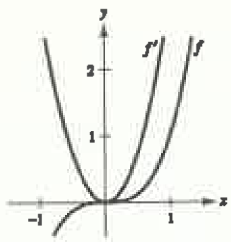
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{x + \Delta x + 1}{x + \Delta x - 1} - \frac{x + 1}{x - 1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x + 1)(x - 1) - (x + \Delta x - 1)(x + 1)}{\Delta x(x + \Delta x - 1)(x - 1)}$$

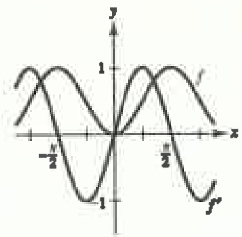
$$= \lim_{\Delta x \to 0} \frac{(x^2 + x\Delta x + x - x - \Delta x - 1) - (x^2 + x\Delta x - x + x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)}$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} = \lim_{\Delta x \to 0} \frac{-2}{(x + \Delta x - 1)(x - 1)} = \frac{-2}{(x - 1)^2}$$

3. (a)



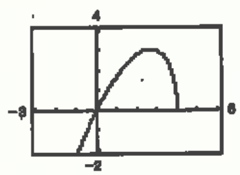
(b)



$f' > 0$ where the slopes of tangent lines to the graph of f are positive.

4. $f(x) = x\sqrt{4 - x}$

(a)



(b) $f'(x) = x\left(\frac{1}{2}\right)(4 - x)^{-1/2}(-1) + (4 - x)^{1/2}$

$$= \frac{1}{2}(4 - x)^{-1/2}[-x + 2(4 - x)]$$

$$= \frac{8 - 3x}{2\sqrt{4 - x}}$$

When $x = 0$, $f'(0) = 2$.

Tangent line: $y = 2x$

(c) $x = 0$

Δx	$f(x + \Delta x)$	$f(x)$	$\frac{f(x + \Delta x) - f(x)}{\Delta x}$
2	2.8284	0	1.4142
1	1.7321	0	1.732
0.5	0.9354	0	1.8708
0.1	0.1975	0	1.9748

(d) $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ represents the slope of the line through $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$. As $\Delta x \to 0$, this quantity approaches $f'(x)$. That is, the numbers in the last column approach $f'(0) = 2$.

5. f is differentiable for all $x \neq 1$.

6. f is differentiable for all $x \neq -3$.

7. $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

8. $f(x) = x^{1/2} - x^{-1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} = \frac{x + 1}{2x^{3/2}}$$

9. $f(x) = 2x - x^{-2}$

$$f'(x) = 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right)$$

$$= \frac{2(x^3 + 1)}{x^3}$$

10. $f(x) = \frac{x + 1}{x - 1}$

$$f'(x) = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}$$

$$= \frac{-2}{(x - 1)^2}$$

11. $g(t) = \frac{2}{3}t^{-2}$

$$g'(x) = \frac{-4}{3}t^{-3} = \frac{-4}{3t^3}$$

12. $h(x) = \frac{2}{9}x^{-2}$

$$h'(x) = \frac{-4}{9}x^{-3} = \frac{-4}{9x^3}$$

13. $f(x) = (1 - x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{1 - x^3}}$$

14. $f(x) = (x^2 - 1)^{1/3}$

$$f'(x) = \frac{1}{3}(x^2 - 1)^{-2/3}(2x)$$

$$= \frac{2x}{3(x^2 - 1)^{2/3}}$$

15. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$$f'(x) = (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x)$$
$$= 2(6x^3 - 9x^2 + 16x - 7)$$

16. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\tfrac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$$
$$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$$
$$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$$

17. $f(x) = \left(x^2 + \dfrac{1}{x}\right)^5$

$$f'(x) = 5\left(x^2 + \frac{1}{x}\right)^4\left(2x - \frac{1}{x^2}\right)$$

18. $h(\theta) = \dfrac{\theta}{(1 - \theta)^3}$

$$h'(\theta) = \frac{(1 - \theta)^3 - \theta[3(1 - \theta)^2(-1)]}{(1 - \theta)^6}$$
$$= \frac{(1 - \theta)^2(1 - \theta + 3\theta)}{(1 - \theta)^6} = \frac{2\theta + 1}{(1 - \theta)^4}$$

19. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2}$$
$$= \frac{-(x^2 + 1)}{(x^2 - 1)^2}$$

20. $f(x) = \dfrac{6x - 5}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(6) - (6x - 5)(2x)}{(x^2 + 1)^2}$$
$$= \frac{2(3 + 5x - 3x^2)}{(x^2 + 1)^2}$$

21. $f(x) = (4 - 3x^2)^{-1}$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

22. $f(x) = 9(3x^2 - 2x)^{-1}$

$$f'(x) = -9(3x^2 - 2x)^{-2}(6x - 2) = \frac{18(1 - 3x)}{(3x^2 - 2x)^2}$$

23. $y = 3\cos(3x + 1)$

$$y' = -9\sin(3x + 1)$$

24. $y = 1 - \cos 2x + 2\cos^2 x$

$$y' = 2\sin 2x - 4\cos x \sin x$$
$$= 2[2\sin x\cos x] - 4\sin x\cos x$$
$$= 0$$

25. $y = \dfrac{1}{2}\csc 2x$

$$y' = \frac{1}{2}(-\csc 2x\cot 2x)(2)$$
$$= -\csc 2x\cot 2x$$

26. $y = \csc 3x + \cot 3x$

$$y' = -3\csc 3x\cot 3x - 3\csc^2 3x$$
$$= -3\csc 3x(\cot 3x + \csc 3x)$$

27. $y = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$

$$y' = \frac{1}{2} - \frac{1}{4}\cos 2x(2)$$
$$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

28. $y = \dfrac{1 + \sin x}{1 - \sin x}$

$$y' = \frac{(1 - \sin x)\cos x - (1 + \sin x)(-\cos x)}{(1 - \sin x)^2}$$
$$= \frac{2\cos x}{(1 - \sin x)^2}$$

29. $y = \dfrac{2}{3}\sin^{3/2} x - \dfrac{2}{7}\sin^{7/2} x$

$$y' = \sin^{1/2} x\cos x - \sin^{5/2} x\cos x$$
$$= (\cos x)\sqrt{\sin x}(1 - \sin^2 x)$$
$$= (\cos^3 x)\sqrt{\sin x}$$

30. $y = \dfrac{\sec^7 x}{7} - \dfrac{\sec^5 x}{5}$

$$y' = \sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x)$$
$$= \sec^5 x \tan x(\sec^2 x - 1)$$
$$= \sec^5 x \tan^3 x$$

31. $y = -x \tan x$

$$y' = -x \sec^2 x - \tan x$$

32. $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

33. $y = \dfrac{\sin x}{x^2}$

$$y' = \frac{(x^2)\cos x - (\sin x)(2x)}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}$$

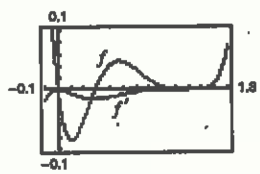
34. $y = \dfrac{\cos(x-1)}{x-1}$

$$y' = \frac{-(x-1)\sin(x-1) - \cos(x-1)(1)}{(x-1)^2}$$
$$= -\frac{1}{(x-1)^2}[(x-1)\sin(x-1) + \cos(x-1)]$$

35. $f(t) = t^2(t-1)^5$

$$f'(t) = t(t-1)^4(7t-2)$$

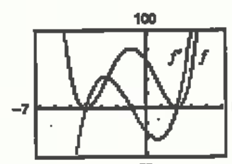
The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



36. $f(x) = (x^2 + 2x - 8)^2$

$$f'(x) = 4(x^3 + 3x^2 - 6x - 8)$$
$$= 4(x-2)(x+1)(x+4)$$

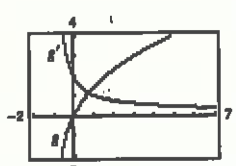
The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



37. $g(x) = 2x(x+1)^{-1/2}$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

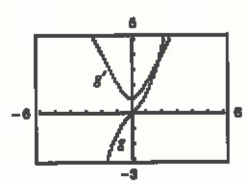
g' does not equal zero for any value of x in the domain. The graph of g has no horizontal tangent lines.



38. $g(x) = x(x^2+1)^{1/2}$

$$g'(x) = \frac{2x^2+1}{\sqrt{x^2+1}}$$

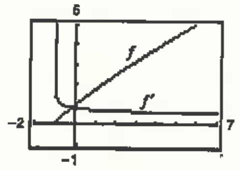
g' does not equal zero for any value of x. The graph of g has no horizontal tangent lines.



39. $f(t) = (t+1)^{1/2}(t+1)^{1/3} = (t+1)^{5/6}$

$$f'(t) = \frac{5}{6(t+1)^{1/6}}$$

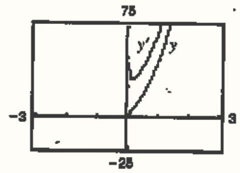
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



40. $y = \sqrt{3x}(x+2)^3$

$$y' = \frac{3(x+2)^2(7x+2)}{2\sqrt{3x}}$$

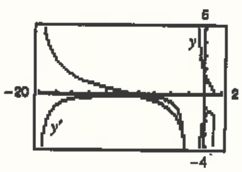
y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



41. $y = \tan\sqrt{1-x}$

$$y' = -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$$

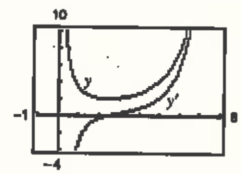
y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



42. $y = 2\csc^3\sqrt{x}$

$$y' = -\frac{3}{\sqrt{x}}\csc^3\sqrt{x}\cot\sqrt{x}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



43. $f(x) = \cos\dfrac{\pi x}{2}$

$$f'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2}$$

$$f'\left(\frac{2}{3}\right) = -\frac{\pi}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{4}$$

44. $f(x) = x + 8x^{-2}$

$f'(x) = 1 - 16x^{-3}$

$$f'(2) = 1 - \frac{16}{2^3} = 1 - 2 = -1$$

45. $y = 2x^2 + \sin 2x$

$y' = 4x + 2\cos 2x$

$y'' = 4 - 4\sin 2x$

46. $y = x^{-1} + \tan x$

$y' = -x^{-2} + \sec^2 x$

$y'' = 2x^{-3} + 2\sec x(\sec x\tan x)$

$\quad = \dfrac{2}{x^3} + 2\sec^2 x\tan x$

47. $f(x) = \cot x$

$f'(x) = -\csc^2 x$

$f'' = -2\csc x(-\csc x \cdot \cot x)$

$\quad = 2\csc^2 x\cot x$

48. $y = \sin^2 x$

$y' = 2\sin x\cos x = \sin 2x$

$y'' = 2\cos 2x$

49. $f(t) = \dfrac{t}{(1-t)^2}$

$$f'(t) = \frac{t+1}{(1-t)^3}$$

$$f''(t) = \frac{2(t+2)}{(1-t)^4}$$

50. $g(x) = \dfrac{6x-5}{x^2+1}$

$$g'(x) = \frac{2(-3x^2+5x+3)}{(x^2+1)^2}$$

$$g''(x) = \frac{2(6x^3-15x^2-18x+5)}{(x^2+1)^3}$$

51. $g(x) = x \tan x$

$$g'(x) = x \sec^2 x + \tan x$$

$$g''(x) = 2 \sec^2 x(x \tan x + 1)$$

52. $h(x) = x\sqrt{x^2 - 1}$

$$h'(x) = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$h''(x) = \frac{x(2x^2 - 3)}{(x^2 - 1)^{3/2}}$$

53.

$$x^2 + 3xy + y^3 = 10$$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

54. $x^2 + 9y^2 - 4x + 3y = 0$

$$2x + 18yy' - 4 + 3y' = 0$$

$$3(6y + 1)y' = 4 - 2x$$

$$y' = \frac{4 - 2x}{3(6y + 1)}$$

55.

$$y\sqrt{x} - x\sqrt{y} = 16$$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

56.

$$y^2 = x^3 - x^2y + xy - y^2$$

$$0 = x^3 - x^2y + xy - 2y^2$$

$$0 = 3x^2 - x^2y' - 2xy + xy' + y - 4yy'$$

$$(x^2 - x + 4y)y' = 3x^2 - 2xy + y$$

$$y' = \frac{3x^2 - 2xy + y}{x^2 - x + 4y}$$

57.

$$x \sin y = y \cos x$$

$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$

$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$

$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

58.

$$\cos(x + y) = x$$

$$-(1 + y') \sin(x + y) = 1$$

$$-y' \sin(x + y) = 1 + \sin(x + y)$$

$$y' = -\frac{1 + \sin(x + y)}{\sin(x + y)}$$

$$= -\csc(x + 1) - 1$$

59. $y = (x + 3)^3$

$$y' = 3(x + 3)^2$$

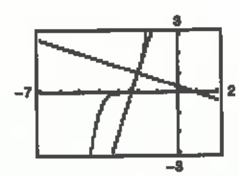
At $(-2, 1)$: $y' = 3$

Tangent line: $y - 1 = 3(x + 2)$

$$3x - y + 7 = 0$$

Normal line: $y - 1 = -\frac{1}{3}(x + 2)$

$$x + 3y - 1 = 0$$



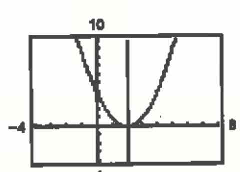
60. $y = (x - 2)^2$

$$y' = 2(x - 2)$$

At $(2, 0)$: $y' = 0$

Tangent line: $y = 0$

Normal line: $x = 2$



61. $x^2 + y^2 = 20$

$2x + 2yy' = 0$

$$y' = -\frac{x}{y}$$

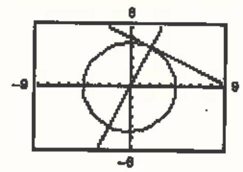
At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line: $y - 4 = 2(x - 2)$

$$2x - y = 0$$



62. $x^2 - y^2 = 16$

$2x - 2yy' = 0$

$$y' = \frac{x}{y}$$

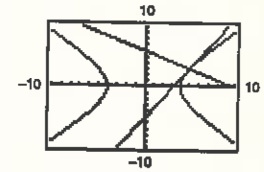
At $(5, 3)$: $y' = \frac{5}{3}$

Tangent line: $y - 3 = \frac{5}{3}(x - 5)$

$$5x - 3y - 16 = 0$$

Normal line: $y - 3 = -\frac{3}{5}(x - 5)$

$$3x + 5y - 30 = 0$$



63. $y = (x - 2)^{2/3}$

$$y' = \frac{2}{3}(x - 2)^{-1/3} = \frac{2}{3\sqrt[3]{x - 2}}$$

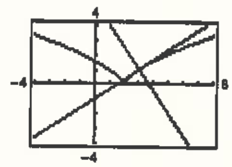
At $(3, 1)$: $y' = \frac{2}{3}$

Tangent line: $y - 1 = \frac{2}{3}(x - 3)$

$$2x - 3y - 3 = 0$$

Normal line: $y - 1 = -\frac{3}{2}(x - 3)$

$$3x + 2y - 11 = 0$$



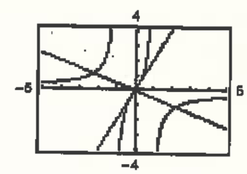
64. $y = \dfrac{2x}{1 - x^2}$

$$y' = \frac{2(1 - x^2) - 2x(-2x)}{(1 - x^2)^2} = \frac{2(x^2 + 1)}{(1 - x^2)^2}$$

At $(0, 0)$: $y' = 2$

Tangent line: $y = 2x$

Normal line: $y = -\frac{1}{2}x$



65. $f(x) = \frac{1}{3}x^3 + x^2 - x - 1$

$f'(x) = x^2 + 2x - 1$

(a) $x^2 + 2x - 1 = -1$

$x(x + 2) = 0$

$(0, -1), \left(-2, \frac{7}{3}\right)$

(b) $x^2 + 2x - 1 = 2$

$(x + 3)(x - 1) = 0$

$(-3, 2), \left(1, -\frac{2}{3}\right)$

(c) $x^2 + 2x - 1 = 0$

$(x + 1)^2 = 2$

$x = -1 \pm \sqrt{2}$

$$\left(-1 + \sqrt{2}, \frac{2\left(1 - 2\sqrt{2}\right)}{3}\right),$$

$$\left(-1 - \sqrt{2}, \frac{2\left(1 + 2\sqrt{2}\right)}{3}\right)$$

66. $f(x) = x^2 + 1$

$f'(x) = 2x$

(a) $2x = -1$

$x = -\frac{1}{2}$

$\left(-\frac{1}{2}, \frac{5}{4}\right)$

(b) $2x = 0$

$x = 0$

$(0, 1)$

(c) $2x = 1$

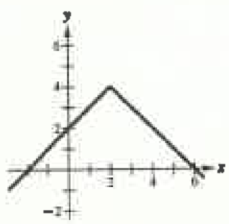
$x = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{5}{4}\right)$

67. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

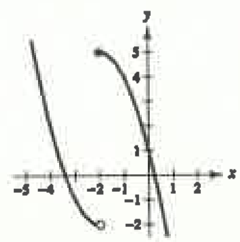
(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.



68. $f(x) = \begin{cases} x^2 + 4x + 2, & \text{if } x < -2 \\ 1 - 4x - x^2, & \text{if } x \geq -2 \end{cases}$

(a) Nonremovable discontinuity at $x = -2$.

(b) Not differentiable at $x = -2$ because the function is discontinuous there.



69.

$$\begin{aligned} y &= 2 \sin x + 3 \cos x \\ y' &= 2 \cos x - 3 \sin x \\ y'' &= -2 \sin x - 3 \cos x \\ y'' + y &= -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x) \\ &= 0 \end{aligned}$$

70.

$$y = \frac{(10 - \cos x)}{x}$$

$$\begin{aligned} xy + \cos x &= 10 \\ xy' + y - \sin x &= 0 \\ xy' &= \sin x - y \\ xy' + y &= (\sin x - y) + y = \sin x \end{aligned}$$

71. $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5 + 2)}{(25 + 30 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

72. $v = \sqrt{2gh} = \sqrt{2(32)h} = 8\sqrt{h}$

$$\frac{dv}{dh} = \frac{4}{\sqrt{h}}$$

(a) When $h = 9$, $\frac{dv}{dh} = \frac{4}{3}$ ft/sec.

(b) When $h = 4$, $\frac{dv}{dh} = 2$ ft/sec.

73. $F = 200\sqrt{T}$

$$F'(T) = \frac{100}{\sqrt{T}}$$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

74. $s = -16t^2 + s_0$

First ball:

$$-16t^2 + 100 = 0$$

$$t = \sqrt{\frac{100}{16}} = \frac{10}{4} = 2.5 \text{ seconds to hit ground}$$

Second ball:

$$-16t^2 + 75 = 0$$

$$t^2 = \sqrt{\frac{75}{16}} = \frac{5\sqrt{3}}{4} \approx 2.165 \text{ seconds to hit ground}$$

Since the second ball was released one second after the first ball, the first ball will hit the ground first. The second ball will hit the ground $3.165 - 2.5 = 0.665$ second later.

75. Assume that the stone is thrown from an initial height of $s_0 = 0$. Thus, the position equation is $s = -16t^2 + v_0t$. The maximum value of s occurs when $ds/dt = 0$ and thus

$$\frac{ds}{dt} = -32t + v_0 = 0$$

$$-32t = -v_0$$

$$t = \frac{v_0}{32}.$$

This means that the maximum height is

$$s = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = \frac{v_0^2}{64}.$$

If s is to attain a value of 49, then

$$\frac{v_0^2}{64} = 49$$

$$v_0^2 = 3136$$

$$v_0 = 56 \text{ ft/sec.}$$

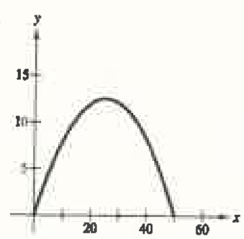
76. $s(t) = -16t^2 + 14{,}400 = 0$

$16t^2 = 14{,}400$

$t = 30$ sec

Since 600 mph $= \frac{1}{6}$ mi/sec, in 30 seconds the bomb will move horizontally $\left(\frac{1}{6}\right)(30) = 5$ miles.

77. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$$0 = x\left(x - \frac{x}{50}\right) \text{ implies } x = 50.$$

(c) Ball reaches maximum height when $x = 25$.

(d)
$$y = x - 0.02x^2$$
$$y' = 1 - 0.04x$$
$$y'(0) = 1$$
$$y'(10) = 0.6$$
$$y'(25) = 0$$
$$y'(30) = -0.2$$
$$y'(50) = -1$$

(e) $y'(25) = 0$

78. $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt} \implies \frac{dx}{dt} = 2\sqrt{x}\frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

79. $y = \sqrt{x}$

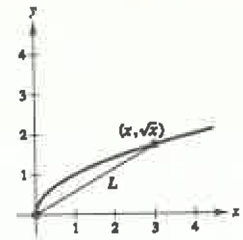
$L^2 = x^2 + y^2$

$\dfrac{dy}{dt} = 2 \text{ units/sec}$

$L^2 = y^4 + y^2$

$2L\dfrac{dL}{dt} = (4y^3 + 2y)\dfrac{dy}{dt}$

$\dfrac{dL}{dt} = \dfrac{4y^3 + 2y}{2L}\dfrac{dy}{dt} = \dfrac{4y^3 + 2y}{L} = \dfrac{(4x + 2)\sqrt{x}}{L}$

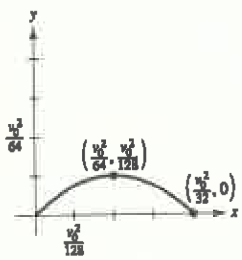


(a) When $x = \dfrac{1}{2}$, $L = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{dL}{dt} = \dfrac{(2 + 2)(1/\sqrt{2})}{\sqrt{3}/2} = \dfrac{8}{\sqrt{6}} = \dfrac{4\sqrt{6}}{3}$ units/sec.

(b) When $x = 1$, $L = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$ and $\dfrac{dL}{dt} = \dfrac{(4 + 2)(1)}{\sqrt{2}} = 3\sqrt{2}$ units/sec.

(c) When $x = 4$, $L = \sqrt{(4)^2 + (2)^2} = 2\sqrt{5}$ and $\dfrac{dL}{dt} = \dfrac{(16 + 2)(2)}{2\sqrt{5}} = \dfrac{18}{\sqrt{5}} = \dfrac{18\sqrt{5}}{5}$ units/sec.

80.



(a) $y = x - \dfrac{32}{v_0^2}x^2 = x\left(1 - \dfrac{32}{v_0^2}x\right)$

$= 0$ if $x = 0$ or $x = \dfrac{v_0^2}{32}$.

Projectile strikes the ground when $x = v_0^2/32$.

Projectile reaches its maximum height at $x = v_0^2/64$. (one-half the distance)

(b) $y' = 1 - \dfrac{64}{v_0^2}x$

When $x = \dfrac{v_0^2}{64}$, $y' = 1 - \dfrac{64}{v_0^2}\left(\dfrac{v_0^2}{64}\right) = 0$.

(d) $v_0 = 70$ ft/sec

Range: $x = \dfrac{v_0^2}{32} = \dfrac{(70)^2}{32} = 153.125$ ft

Maximum height: $y = \dfrac{v_0^2}{128} = \dfrac{(70)^2}{128} \approx 38.28$ ft

(c) $y = x - \dfrac{32}{v_0^2}x^2 = x\left(1 - \dfrac{32}{v_0^2}x\right) = 0$

when $x = 0$ and $x = x_0^2/32$. Therefore, the range is $x = v_0^2/32$. When the initial velocity is doubled the range is

$$x = \frac{(2v_0)^2}{32} = \frac{4v_0^2}{32}$$

or four times the initial range. From part (a), the maximum height occurs when $x = v_0^2/64$. The maximum height is

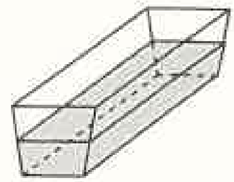
$$y\left(\frac{v_0^2}{64}\right) = \frac{v_0^2}{64} - \frac{32}{v_0^2}\left(\frac{v_0^2}{64}\right)^2 = \frac{v_0^2}{64} - \frac{v_0^2}{128} = \frac{v_0^2}{128}.$$

If the initial velocity is doubled, the maximum height is

$$y\left[\frac{(2v_0)^2}{64}\right] = \frac{(2v_0)^2}{128} = 4\left(\frac{v_0^2}{128}\right)$$

or four times the original maximum height.

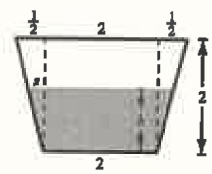
81. $\dfrac{s}{h} = \dfrac{1/2}{2}$



$s = \dfrac{1}{4}h$

$\dfrac{dV}{dt} = 1$

Width of water at depth h: $w = 2 + 2s = 2 + 2\left(\dfrac{1}{4}h\right) = \dfrac{4+h}{2}$



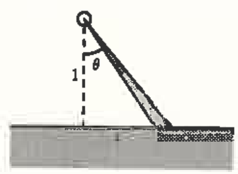
$$V = \frac{5}{2}\left(2 + \frac{4+h}{2}\right)h = \frac{5}{4}(8+h)h$$

$$\frac{dV}{dt} = \frac{5}{2}(4+h)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4+h)}$$

When $h = 1$, $\dfrac{dh}{dt} = \dfrac{2}{25}$ m/min.

82. $\tan\theta = x$



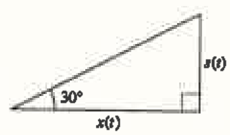
$$\frac{d\theta}{dt} = 3(2\pi) \text{ rad/min}$$

$$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = (\tan^2\theta + 1)(6\pi) = 6\pi(x^2+1)$$

When $x = \dfrac{1}{2}$, $\dfrac{dx}{dt} = 6\pi\left(\dfrac{1}{4} + 1\right) = \dfrac{15\pi}{2}$ km/min $= 450\pi$ km/hr.

83. $s(t) = 60 - 4.9t^2$



$s'(t) = -9.8t$

$s = 35 = 60 - 4.9t^2$

$$4.9t^2 = 25$$

$$t = \frac{5}{\sqrt{4.9}}$$

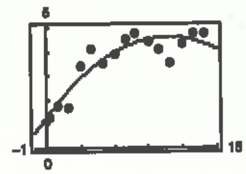
$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

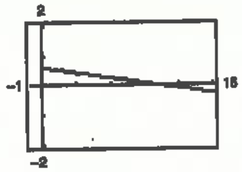
$$\frac{dx}{dt} = \sqrt{3}\frac{ds}{dt} = \sqrt{3}(-9.8)\frac{5}{\sqrt{4.9}} \approx -38.34 \text{ m/sec}$$

84. (a) $S = -0.0285876t^2 + 0.61594t + 1.296176$

(b)



(c) $\dfrac{ds}{dt} = -0.057175t + 0.61594$



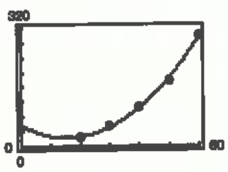
(d) Yes, the derivative is negative from $t = 9$ to $t = 14$. Perhaps the market was saturated.

(e) No, some years were worse than predicted by the model.

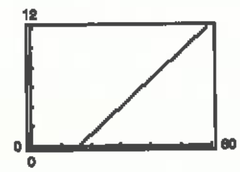
(f) Sales were increasing most rapidly in 1980–81 according to the model.

85. (a) $y = 0.14x^2 - 4.43x + 58.4$

(b)



(c)



(d) If $x = 65$, $y \approx 362$ feet.

(e) As the speed increases, the stopping distance increases at an increasing rate.

86. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$a(t) = v'(t) = 2$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$

(d) $x(t) = 0$ for $t = 1, 2$.

$|v(1)| = |2(1) - 3| = 1$

$|v(2)| = |2(2) - 3| = 1$

The speed is 1 when the position is 0.