CHAPTER 3
Applications of Differentiation

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Section 3.1 Extrema on an Interval

Solutions to Exercises

1. $f(x) = \dfrac{x^2}{x^2+4}$

$f'(x) = \dfrac{(x^2+4)(2x) - (x^2)(2x)}{(x^2+4)^2} = \dfrac{8x}{(x^2+4)^2}$

$f'(0) = 0$

2. $f(x) = \cos\dfrac{\pi x}{2}$

$f'(x) = -\dfrac{\pi}{2}\sin\dfrac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

3. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3}$

$f'(4) = 0$

4. $f(x) = -3x\sqrt{x+1}$

$f'(x) = -3x\left[\dfrac{1}{2}(x+1)^{-1/2}\right] + \sqrt{x+1}(-3)$

$= -\dfrac{3}{2}(x+1)^{-1/2}[x + 2(x+1)]$

$= -\dfrac{3}{2}(x+1)^{-1/2}(3x+2)$

$f'\left(-\dfrac{2}{3}\right) = 0$

5. $f(x) = (x+2)^{2/3}$

$f'(x) = \dfrac{2}{3}(x+2)^{-1/3}$

$f'(-2)$ is undefined.

6. Using the limit definition of the derivative,

$\lim_{x\to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \dfrac{(4 - |x|) - 4}{x} = 1$

$\lim_{x\to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \dfrac{(4 - |x|) - 4}{x - 0} = -1$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

7. $f(x) = x^2(x-3) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

Critical numbers: $x = 0, x = 2$

8. $g(x) = x^2(x^2-4) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x = 4x(x^2-2)$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

9. $g(t) = t\sqrt{4-t}$

$g'(t) = t\left[\dfrac{1}{2}(4-t)^{-1/2}(-1)\right] + (4-t)^{1/2}$

$= \dfrac{1}{2}(4-t)^{-1/2}[-t + 2(4-t)]$

$= \dfrac{8-3t}{2\sqrt{4-t}}$

Critical numbers: $t = 4, t = \frac{8}{3}$

10. $f(x) = \dfrac{4x}{x^2+1}$

$f'(x) = \dfrac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \dfrac{4(1-x^2)}{(x^2+1)^2}$

Critical numbers: $x = \pm 1$

11. $h(x) = \sin^2 x + \cos x, 0 \le x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

Critical numbers: $x = 0, x = \frac{\pi}{3}, x = \pi, x = \frac{5\pi}{3}$

12. $f(\theta) = 2 \sec \theta + \tan \theta, 0 \le \theta < 2\pi$

$f'(\theta) = 2 \sec \theta \tan \theta + \sec^2 \theta$

$= \sec \theta(2 \tan \theta + \sec \theta)$

$= \sec \theta\left[2\left(\frac{\sin \theta}{\cos \theta}\right) + \frac{1}{\cos \theta}\right]$

$= \sec^2 \theta(2 \sin \theta + 1)$

Critical numbers: $\theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$

13. $f(x) = 2(3 - x), [-1, 2]$

$f'(x) = -2 \Rightarrow$ No critical numbers

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

14. $f(x) = \frac{2x + 5}{3}, [0, 5]$

$f'(x) = \frac{2}{3} \Rightarrow$ No critical numbers

Left endpoint: $\left(0, \frac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

15. $f(x) = -x^2 + 3x, [0, 3]$

$f'(x) = -2x + 3$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3}{2}, \frac{9}{4}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

16. $f(x) = x^2 + 2x - 4, [-1, 1]$

$f'(x) = 2x + 2 = 2(x + 1)$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

17. $f(x) = x^3 - 3x^2, [-1, 3]$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Left endpoint: $(-1, -4)$ Minimum

Critical number: $(0, 0)$ Maximum

Critical number: $(2, -4)$ Minimum

Right endpoint: $(3, 0)$ Maximum

18. $f(x) = x^3 - 12x, [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

Note: $x = -2$ is not in the interval.

19. $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

20. $g(x) = \sqrt[3]{x}, [-1, 1]$

$g'(x) = \frac{1}{3x^{2/3}}$

Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

21. $h(t) = 4 - |t - 4|, [1, 6]$

From the graph of the function on the interval $[1, 6]$ you can determine the following.

Left endpoint: $(1, 1)$ Minimum

Critical number: $(4, 4)$ Maximum

Right endpoint: $(6, 2)$

22. $g(t) = \dfrac{t^2}{t^2 + 3}, [-1, 1]$

$g'(t) = \dfrac{6t}{(t^2 + 3)^2}$

Left endpoint: $\left(-1, \dfrac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \dfrac{1}{4}\right)$ Maximum

23. $h(s) = \dfrac{1}{s - 2}, [0, 1]$

$h'(s) = \dfrac{-1}{(s - 2)^2}$

Left endpoint: $\left(0, -\dfrac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

24. $h(t) = \dfrac{t}{t - 2}, [3, 5]$

$h'(t) = \dfrac{-2}{(t - 2)^2}$

Left endpoint: $(3, 3)$ Maximum

Right endpoint: $\left(5, \dfrac{5}{3}\right)$ Minimum

25. $f(x) = \cos \pi x, \left[0, \dfrac{1}{6}\right]$

$f'(x) = -\pi \sin \pi x$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\dfrac{1}{6}, \dfrac{\sqrt{3}}{2}\right)$ Minimum

26. $g(x) = \csc x, \left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$g'(x) = -\csc x \cot x$

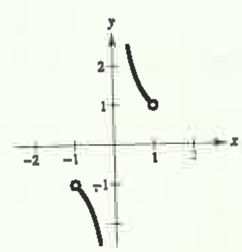
Left endpoint: $\left(\dfrac{\pi}{6}, 2\right)$ Maximum

Right endpoint: $\left(\dfrac{\pi}{3}, \dfrac{2\sqrt{3}}{3}\right)$ Minimum

27. $f(x) = \tan x$

f is continuous on $[0, \pi/4]$ but not on $[0, \pi]$. $\displaystyle\lim_{x \to \pi/2^-} \tan x = \infty$.

28. Let $f(x) = 1/x$. f is continuous on $(0, 1)$ but does not have a maximum. f is also continuous on $(-1, 0)$ but does not have a minimum. This can occur if one of the endpoints is an infinite discontinuity.



29. (a) Yes

(b) No

30. (a) No

(b) Yes

31. (a) No

(b) Yes

32. (a) No

(b) Yes

33. (a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

34. (a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

35. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

36. (a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

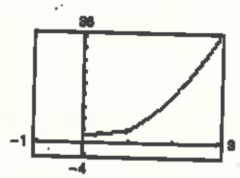
(c) Maximum: $(0, 2)$

(d) Maximum: $\left(1, \sqrt{3}\right)$

37. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$

Left endpoint: $(0, 2)$ Minimum

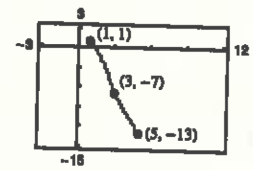
Right endpoint: $(3, 36)$ Maximum



38. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$

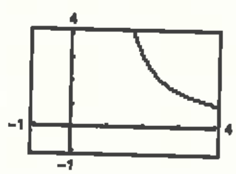
Left endpoint: $(1, 1)$ Maximum

Right endpoint: $(5, -13)$ Minimum



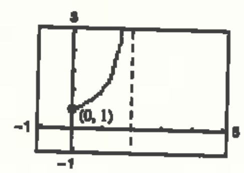
39. $f(x) = \dfrac{3}{x-1}, [1, 4]$

Right endpoint: $(4, 1)$ Minimum

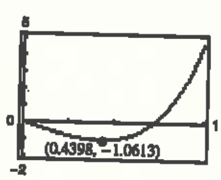


40. $f(x) = \dfrac{2}{2-x}, [0, 2)$

Left endpoint: $(0, 1)$ Minimum



41. (a)



Maximum: $(1, 4.7)$

Minimum: $(0.4398, -1.0613)$

(b)

$$f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

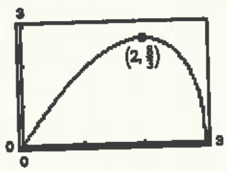
$$f(0) = 0$$

$f(1) = 4.7$ Maximum

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: $(0.4398, -1.0613)$

42. (a)



Maximum: $\left(2, \frac{8}{3}\right)$

Minimum: $(0, 0), (3, 0)$

(b) $f(x) = \frac{4}{3}x\sqrt{3 - x}, [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right]$$

$$= \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)[-x + 2(3 - x)]$$

$$= \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Critical number: $x = 2$

$f(0) = 0$ Minimum

$f(3) = 0$ Minimum

$f(2) = \frac{8}{3}$

Maximum: $\left(2, \frac{8}{3}\right)$

43. $f(x) = (1 + x^3)^{1/2}, [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, we have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1 \approx 0.732$.

$\left|f''\left(\sqrt[3]{-10 + \sqrt{108}}\right)\right| \approx 1.47$ is the maximum value.

44. $f(x) = \frac{1}{x^2 + 1}, \left[\frac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

Setting $f''' = 0$, we have $x = 0, \pm 1$.

$|f''(1)| = \frac{1}{2}$ is the maximum value.

45. $f(x) = (x + 1)^{2/3}, [0, 2]$

$$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$$

$$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$$

$$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$$

$|f^{(4)}(0)| = \frac{56}{81}$ is the maximum value.

46. $f(x) = \frac{1}{x^2 + 1}, [-1, 1]$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4} \text{ (See Exercise 44.)}$$

$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

$$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$$

$|f^{(4)}(0)| = 24$ is the maximum value.

47. $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. P is decreasing for $I > 12$.

48. $C = 2x + \dfrac{300{,}000}{x}, 1 \le x \le 300$

$C(1) = 300{,}002$

$C(300) = 1600$

$C' = 2 - \dfrac{300{,}000}{x^2} = 0$

$2x^2 = 300{,}000$

$x^2 = 150{,}000$

$x = 100\sqrt{15} \approx 387 > 300$ (outside of interval)

C is minimized when $x = 300$ units.

Yes, if $1 \le x \le 400$, then $x = 387$ would minimize C.

49. $x = \dfrac{v^2 \sin 2\theta}{32}, \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

$\dfrac{d\theta}{dt}$ is constant.

$\dfrac{dx}{dt} = \dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$ (by the Chain Rule)

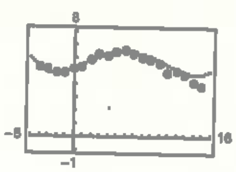
$= \dfrac{v^2 \cos 2\theta}{16}\dfrac{d\theta}{dt}$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for dx/dt and $\theta = \pi/2$ indicates a maximum for dx/dt. This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. Thus, the lawn farthest from the spinkler gets the most water.

50. (a) Using a graphing utility, you obtain

$y = 0.0003674t^4 - 0.009176t^3 + 0.03251t^2 + 0.2630t + 5.1669.$

(b)



(c) Maximum is 6.41% for $t = 5.8$.

Minimum is 4.1% for $t = 15.0$.

51. $S = 6hs + \dfrac{3s^2}{2}\left(\dfrac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2}$

$\dfrac{dS}{d\theta} = \dfrac{3s^2}{2}\left(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta\right)$

$= \dfrac{3s^2}{2}\csc\theta\left(-\sqrt{3}\cot\theta + \csc\theta\right) = 0$

$\csc\theta = \sqrt{3}\cot\theta$

$\sec\theta = \sqrt{3}$

$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radians

$S\left(\dfrac{\pi}{6}\right) = 6hs + \dfrac{3s^2}{2}(\sqrt{3})$

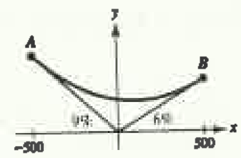
$S\left(\dfrac{\pi}{2}\right) = 6hs + \dfrac{3s^2}{2}(\sqrt{3})$

$S\left(\operatorname{arcsec}\sqrt{3}\right) = 6hs + \dfrac{3s^2}{2}(\sqrt{2})$

S is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radians.

52. (a) $y = ax^2 + bx + c$

$y' = 2ax + b$



The coordinates of B are $(500, 30)$, and those of A are $(-500, 45)$. From the slopes at A and B,

$$-1000a + b = -0.09$$

$$1000a + b = 0.06.$$

Solving these two equations, you obtain $a = 3/40000$ and $b = -3/200$. From the points $(500, 30)$ and $(-500, 45)$, you obtain

$$30 = \frac{3}{40000}500^2 + 500\left(\frac{-3}{200}\right) + c$$

$$45 = \frac{3}{40000}500^2 - 500\left(\frac{-3}{200}\right) + c.$$

In both cases, $c = 18.75 = \frac{75}{4}$. Thus,

$$y = \frac{3}{40000}x^2 - \frac{3}{200}x + \frac{17}{4}.$$

(b)

x	−500	−400	−300	−200	−100	0	100	200	300	400	500
d	0	.75	3	6.75	12	18.75	12	6.75	3	.75	0

For $-500 \le x \le 0$, $d = (ax^2 + bx + c) - (-0.09x)$.

For $0 \le x \le 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(c) The lowest point on the highway is $(100, 18)$, which is not directly over the point where the two hillsides come together.

53. True. See Exercise 17.

54. True. This is stated in the Extreme Value Theorem.

55. True.

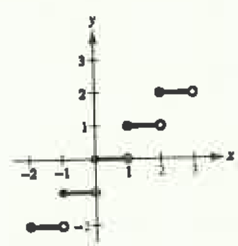
56. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f.

$$g(x) = f(x - k)$$
$$= (x - k)^2$$

$x = k$ is a critical number of g.

57. $f(x) = [\![x]\!]$

The derivative of f is undefined at every integer and is zero at any noninteger real number. All real numbers are critical numbers.



Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ since f is not differentiable at $x = 1$.

2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ since f is not continuous at $x = 2\pi$.

3. $f(x) = x^2 - 2x, [0, 2]$

$f(0) = f(2) = 0$

f is continuous on $[0, 2]$. f is differentiable on $(0, 2)$. Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Longrightarrow x = 1$$

c value: 1

4. $f(x) = x^2 - 3x + 2, [1, 2]$

$f(1) = f(2) = 0$

f is continuous on $[1, 2]$. f is differentiable on $(1, 2)$. Rolle's Theorem applies.

$$f'(x) = 2x - 3$$

$$2x - 3 = 0 \Longrightarrow x = \tfrac{3}{2} = 1.5$$

c value: $\frac{3}{2} = 1.5$

5. $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = f(3) = 0$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$. Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Longrightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}.$$

6. $f(x) = (x - 3)(x + 1)^2, [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$\begin{aligned} f'(x) &= (x - 3)(2)(x + 1) + (x + 1)^2 \\ &= (x + 1)[2x - 6 + x + 1] \\ &= (x + 1)(3x - 5) \end{aligned}$$

c value: $\frac{5}{3}$

7. $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = f(8) = 3$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.

8. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ since $f'(3)$ does not exist. Rolle's Theorem does not apply.

9. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. (**Note:** The discontinuity, $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

c value: $-2+\sqrt{5}$

10. $f(x) = \dfrac{x^2 - 1}{x}$, $[-1, 1]$

$f(-1) = f(1) = 0$

f is not continuous on $[-1, 1]$ since $f(0)$ does not exist. Rolle's Theorem does not apply.

11. $f(x) = \sin x$, $[0, 2\pi]$

$f(0) = f(2\pi) = 0$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = \cos x$

c values: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

12. $f(x) = \cos x$, $[0, 2\pi]$

$f(0) = f(2\pi) = 1$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = -\sin x$

c value: π

13. $f(x) = \sin 2x$, $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$f\left(\dfrac{\pi}{6}\right) = f\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

f is continuous on $[\pi/6, \pi/3]$. f is differentiable on $(\pi/6, \pi/3)$. Rolle's Theorem applies.

$$f'(x) = 2\cos 2x$$

$$2\cos 2x = 0$$

$$x = \frac{\pi}{4}$$

c value: $\dfrac{\pi}{4}$

14. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x$, $\left[0, \dfrac{\pi}{6}\right]$

$f(0) = f\left(\dfrac{\pi}{6}\right) = 0$

f is continuous on $[0, \pi/6]$. f is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8\sin x\cos x = 0$$

$$\frac{6}{\pi} = 8\sin x\cos x$$

$$\frac{3}{4\pi} = \frac{1}{2}\sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2}\arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

c value: 0.2489

15. $f(x) = \tan x$, $[0, \pi]$

$f(0) = f(\pi) = 0$

f is not continuous on $[0, \pi]$ since $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = \sec x, \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$

f is continuous on $[-\pi/4, \pi/4]$. f is differentiable on $(-\pi/4, \pi/4)$. Rolle's Theorem applies.

$$f'(x) = \sec x \tan x$$

$$\sec x \tan x = 0$$

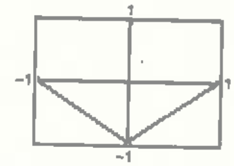
$$x = 0$$

c value: 0

17. $f(x) = |x| - 1, [-1, 1]$

$f(-1) = f(1) = 0$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.



18. $f(x) = x - x^{1/3}, [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$. (**Note:** f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

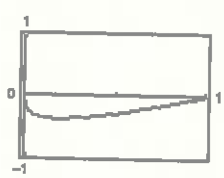
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

c value: $\frac{\sqrt{3}}{9} \approx 0.1925$



19. $f(x) = 4x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$

$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$

f is continuous on $[-1/4, 1/4]$. f is differentiable on $(-1/4, 1/4)$. Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

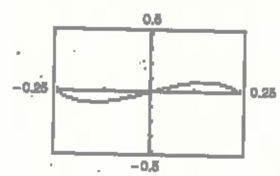
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm\frac{2}{\sqrt{\pi}}$$

$$x = \pm\frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm\frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

c values: ± 0.1533 radian



20. $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}, \; [-1, 0]$

$f(-1) = f(0) = 0$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$. Rolle's Theorem applies.

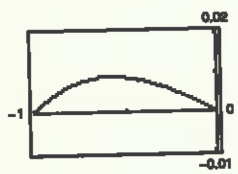
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi} \quad \text{[Value needed in } (-1, 0).]$$

$$\approx -0.5756 \text{ radian}$$

c value: -0.5756



21. $f(t) = -16t^2 + 48t + 32$

(a) $f(1) = f(2) = 64$

(b) $v = f'(t)$ must be 0 at some time in $[1, 2]$.

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

22. $C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$

(a) $C(3) = C(6) = \frac{25}{3}$

(b) $$C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

In the interval $[3, 6]$: $c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098$.

23. No. Let $f(x) = x^2$ on $[-1, 2]$.

$f'(x) = 2x$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

24. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b).

(a) $g(x) = f(x) + k$

$g(a) = g(b) = f(a) + k$

$g'(x) = f'(x) \Rightarrow g'(c) = 0$

Interval: $[a, b]$

Critical number of g: c

(b) $g(x) = f(x - k)$

$g(a + k) = g(b + k) = f(a)$

$g'(x) = f'(x - k)$

$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of g: $c + k$

(c) $g(x) = f(kx)$

$g\left(\frac{a}{k}\right) = g\left(\frac{b}{k}\right) = f(a)$

$g'(x) = kf'(kx)$

$g'\left(\frac{c}{k}\right) = kf'(c) = 0$

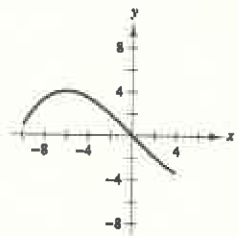
Interval: $\left[\frac{a}{k}, \frac{b}{k}\right]$

Critical number of g: $\frac{c}{k}$

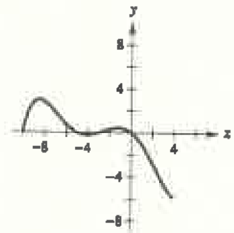
25. (a) f is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0,\ f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

(b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c)



(d)

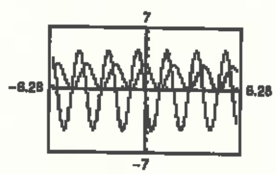


(e) No, f' did not have to be continuous on $[-10, 4]$.

26. $f(x) = 3\cos^2\left(\dfrac{\pi x}{2}\right)$, $f'(x) = 6\cos\left(\dfrac{\pi x}{2}\right)\left(-\sin\left(\dfrac{\pi x}{2}\right)\right)\left(\dfrac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a)



(b) f and f' are both continuous on the entire real line.

(c) Since $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Since $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

(d) $\displaystyle\lim_{x\to 3^-} f'(x) = 0$

$\displaystyle\lim_{x\to 3^+} f'(x) = 0$

27. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$ when $x = -\frac{1}{2}$. Therefore,

$$c = -\frac{1}{2}.$$

28. $f(x) = x(x^2 - x - 2)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = -1$$

$$f'(x) = 3x^2 - 2x - 2 = -1$$

$$(3x + 1)(x - 1) = 0$$

$$c = -\frac{1}{3}$$

29. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

30. $f(x) = (x + 1)/x$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - (1/2)} = \frac{(3/2) - 3}{3/2} = -1$$

$$f'(x) = \frac{-1}{x^2} = -1$$

$$x^2 = 1$$

$$c = 1$$

31. $f(x) = \sqrt{x-2}$ is continuous on $[2, 6]$ and differentiable on $(2, 6)$.

$$\frac{f(6)-f(2)}{6-2} = \frac{2-0}{4} = \frac{1}{2}$$

$$f'(x) = \frac{1}{2\sqrt{x-2}} = \frac{1}{2}$$

$$\sqrt{x-2} = 1$$

$$c = 3$$

32. $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1)-f(0)}{1-0} = \frac{1-0}{1} = 1$$

$$f'(x) = 3x^2 = 1$$

$$x = \pm\frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \dfrac{\sqrt{3}}{3}$.

33. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi)-f(0)}{\pi-0} = \frac{0-0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$c = \frac{\pi}{2}$$

34. $f(x) = 2\sin x + \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi)-f(0)}{\pi-0} = \frac{0-0}{\pi} = 0$$

$$f'(x) = 2\cos x + 2\cos 2x = 0$$

$$2[\cos x + 2\cos^2 x - 1] = 0$$

$$2(2\cos x - 1)(\cos x + 1) = 0$$

$$\cos x = \frac{1}{2}$$

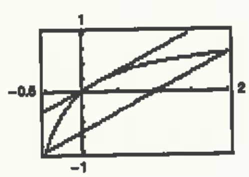
$$\cos x = -1$$

$$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

In the interval $(0, \pi)$: $c = \dfrac{\pi}{3}$.

35. $f(x) = \dfrac{x}{x+1}$ on $\left[-\dfrac{1}{2}, 2\right]$.

(a)



(b) Secant line:

$$\text{slope} = \frac{f(2)-f(-1/2)}{2-(-1/2)} = \frac{2/3-(-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x-2)$$

$$3y - 2 = 2x - 4$$

$$3y - 2x + 2 = 0$$

(c) $f'(x) = \dfrac{1}{(x+1)^2} = \dfrac{2}{3}$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$, $c = -1 + (\sqrt{6}/2)$.

$$f(c) = \frac{-1+(\sqrt{6}/2)}{[-1+(\sqrt{6}/2)]+1} = \frac{-2+\sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

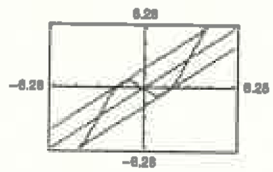
Tangent line: $y - 1 + \dfrac{2}{\sqrt{6}} = \dfrac{2}{3}\left(x - \dfrac{\sqrt{6}}{2} + 1\right)$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$3y - 2x - 5 + 2\sqrt{6} = 0$$

36. $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

(a)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$\cos x = 0$

$$c = \pm\frac{\pi}{2}, \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\frac{\pi}{2} - 2\right) = 1\left(x - \frac{\pi}{2}\right)$

$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

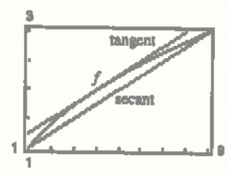
$$y = x + 2$$

37. $f(x) = \sqrt{x}$, $[1, 9]$

$(1, 1), (9, 3)$

$$m = \frac{3 - 1}{9 - 1} = \frac{1}{4}$$

(a)



(b) Secant line: $y - 1 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c) $f'(x) = \dfrac{1}{2\sqrt{x}}$

$$\frac{f(9) - f(1)}{9 - 1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$

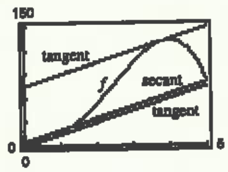
$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

38. $f(x) = -x^4 + 4x^3 + 8x^2 + 5,\ (0, 5), (5, 80)$

$$m = \frac{80 - 5}{5 - 0} = 15$$

(a)



(b) Secant line: $y - 5 = 15(x - 0)$

$$0 = 15x - y + 5$$

$$f'(x) = -4x^3 + 12x^2 + 16x$$

$$\frac{f(5) - f(1)}{5 - 1} = 15$$

$$-4c^3 + 12c^2 + 16c = 15$$

$$0 = 4c^3 - 12c^2 - 16c + 15$$

$$c \approx 0.67 \text{ or } c \approx 3.79$$

(c) First tangent line: $y - f(c) = m(x - c)$

$$y - 9.59 = 15(x - 0.67)$$

$$0 = 15x - y - 0.46$$

Second tangent line: $y - f(c) = m(x - c)$

$$y - 131.35 = 15(x - 3.79)$$

$$0 = 15x - y + 74.5$$

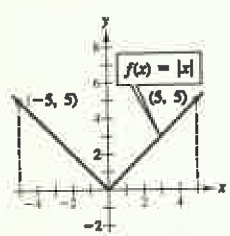
39. $f(x) = \dfrac{1}{x - 3},\ [0, 6]$

f has a discontinuity at $x = 3$.

40. $f(x) = |x - 3|,\ [0, 6]$

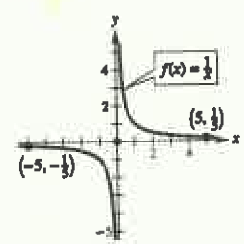
f is not differentiable at $x = 3$.

41. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem.
$\Rightarrow f$ is not differentiable on $(-5, 5)$.
Example: $f(x) = |x|$



42. f is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



43. $s(t) = -4.9t^2 + 500$

(a) $V_{avg} = \dfrac{s(3) - s(0)}{3 - 0} = \dfrac{455.9 - 500}{3} = -14.7$ m/sec

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

44. $S(t) = 200\left(5 - \dfrac{9}{2+t}\right)$

(a) $\dfrac{S(12) - S(0)}{12 - 0} = \dfrac{200[5 - (9/14)] - 200[5 - (9/2)]}{12} = \dfrac{450}{7}$

(b) $S'(t) = 200\left(\dfrac{9}{(2+t)^2}\right) = \dfrac{450}{7}$

$$\frac{1}{(2+t)^2} = \frac{1}{28}$$

$$2 + t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

45. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0, $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

46. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/hr}.$$

By the Mean Value Theorem, there exists a time to, $0 < t_0 < 5$, such that $T'(t_0) = -222$.

47. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

48. False. f must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}$$

49. True. A polynomial is continuous and differentiable everywhere.

50. True

51. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2. Then by Rolle's Theorem, since $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, since $n > 0$, $a > 0$. Therefore, $p(x)$ cannot have two real roots.

52. Suppose $f(x)$ is not constant on (a, b). Then there exists x_1 and x_2 in (a, b) such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists c in (a, b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b).

53. If $p(x) = Ax^2 + Bx + C$, then

$$p'(x) = 2Ax + B = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$$

$$= \frac{A(b^2 - a^2) + B(b - a)}{b - a}$$

$$= \frac{(b - a)[A(b + a) + B]}{b - a}$$

$$= A(b + a) + B.$$

Thus, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

54. Suppose $f(x)$ has two fixed points c_1 and c_2. Then, by the Mean Value Theorem, there exists c such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x.

55. $f(x) = \frac{1}{2}\cos x$ is differentiable on $(-\infty, \infty)$.

$f'(x) = -\frac{1}{2}\sin x$

$-\frac{1}{2} \le f'(x) \le \frac{1}{2} \Rightarrow f'(x) < 1$ for all real numbers.

Thus, from Exercise 54, f has, at most, one fixed point. $(x \approx 0.4502)$

56. Let $f(x) = \cos x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c||b - a|$$

$$|\cos b - \cos a| \le |b - a| \text{ since } |-\sin c| \le 1.$$

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. $f(x) = x^2 - 6x + 8$

   Increasing on: $(3, \infty)$

   Decreasing on: $(-\infty, 3)$

2. $y = -(x + 1)^2$

   Increasing on: $(-\infty, -1)$

   Decreasing on: $(-1, \infty)$

3. $y = \dfrac{x^3}{4} - 3x$

   Increasing on: $(-\infty, -2), (2, \infty)$

   Decreasing on: $(-2, 2)$

4. $f(x) = x^4 - 2x^2$

   Increasing on: $(-1, 0), (1, \infty)$

   Decreasing on: $(-\infty, -1), (0, 1)$

5. $f(x) = \dfrac{1}{x^2}$

   Increasing on: $(-\infty, 0)$

   Decreasing on: $(0, \infty)$

6. $y = \dfrac{x^2}{x + 1}$

   Increasing on: $(-\infty, -2), (0, \infty)$

   Decreasing on: $(-2, -1), (-1, 0)$

7. $f(x) = -2x^2 + 4x + 3$

   $f'(x) = -4x + 4 = 0$

   Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

   Increasing on: $(-\infty, 1)$

   Decreasing on: $(1, \infty)$

   Relative maximum: $(1, 5)$

8. $f(x) = x^2 + 8x + 10$

   $f'(x) = 2x + 8 = 0$

   Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

   Increasing on: $(-4, \infty)$

   Decreasing on: $(-\infty, -4)$

   Relative minimum: $(-4, -6)$

9. $f(x) = x^2 - 6x$

   $f'(x) = 2x - 6 = 0$

   Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

   Increasing on: $(3, \infty)$

   Decreasing on: $(-\infty, 3)$

   Relative minimum: $(3, -9)$

10. $f(x) = (x - 1)^2(x + 2)$

$f'(x) = (x - 1)^2(1) + (x + 2)(2)(x - 1) = (x - 1)[(x - 1) + 2(x + 2)] = 3(x - 1)(x + 1) = 0$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

Relative maximum: $(-1, 4)$

Relative minimum: $(1, 0)$

11. $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$

Critical numbers: $x = -2, 1$

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$

Decreasing on: $(-2, 1)$

Relative maximum: $(-2, 20)$

Relative minimum: $(1, -7)$

12. $f(x) = (x - 3)^3$

$f'(x) = 3(x - 3)^2 = 0$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

13. $f(x) = \dfrac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

Relative maximum: $\left(-1, \frac{4}{5}\right)$

Relative minimum: $\left(1, -\frac{4}{5}\right)$

14. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

Relative minimum: $(2, -44)$

15. $f(x) = x^{1/3} + 1$

$f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

16. $f(x) = x^{2/3}(x - 5) = x^{5/3} - 5x^{2/3}$

$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x - 2)$

Critical numbers: $x = 0, 2$

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, 0), (2, \infty)$

Decreasing on: $(0, 2)$

Relative maximum: $(0, 0)$

Relative minimum: $\left(2, -3\sqrt[3]{4}\right)$

17. $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x-5}{|x-5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

Critical number: $x = 5$

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 5)$

Decreasing on: $(5, \infty)$

Relative maximum: $(5, 5)$

18. $f(x) = |x + 3| - 1$

$$f'(x) = \frac{x+3}{|x+3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$$

Critical number: $x = -3$

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

Relative minimum: $(-3, -1)$

19. $f(x) = \dfrac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

Relative maximum: $(0, 0)$

20. $f(x) = \dfrac{x+3}{x^2} = \dfrac{1}{x} + \dfrac{3}{x^2}$

$$f'(x) = -\frac{1}{x^2} - \frac{6}{x^3} = \frac{-(x+6)}{x^3}$$

Critical number: $x = -6$

Discontinuity: $x = 0$

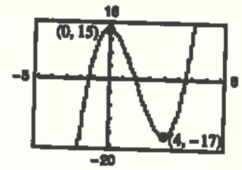
Test intervals:	$-\infty < x < -6$	$-6 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-6, 0)$

Decreasing on: $(-\infty, -6), (0, \infty)$

Relative minimum: $\left(-6, -\dfrac{1}{12}\right)$

21. $f(x) = x^3 - 6x^2 + 15$



$f'(x) = 3x^2 - 12x = 3x(x - 4) = 0$

Critical numbers: $x = 0, 4$

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

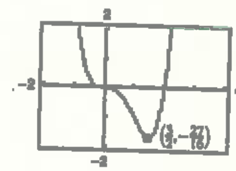
Increasing on: $(-\infty, 0), (4, \infty)$

Decreasing on: $(0, 4)$

Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

22. $f(x) = x^4 - 2x^3$



$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3) = 0$

Crititcal numbers: $x = 0, \frac{3}{2}$

Test intervals:	$-\infty < x < 0$	$0 < x < \frac{3}{2}$	$\frac{3}{2} < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Decreasing	Increasing

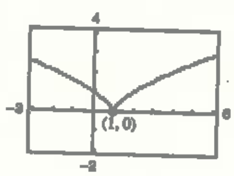
Decreasing on: $\left(-\infty, \frac{3}{2}\right)$

Increasing on: $\left(\frac{3}{2}, \infty\right)$

Relative minimum: $\left(\frac{3}{2}, -\frac{27}{16}\right)$

23. $f(x) = (x - 1)^{2/3}$

$f'(x) = \dfrac{2}{3(x - 1)^{1/3}}$



Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

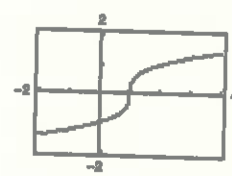
Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

Relative minimum: $(1, 0)$

24. $f(x) = (x - 1)^{1/3}$

$f'(x) = \dfrac{1}{3(x - 1)^{2/3}}$



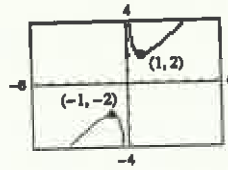
Critical number:
$x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

25. $f(x) = x + \dfrac{1}{x}$



$f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2}$

Critical numbers: $x = -1, 1$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

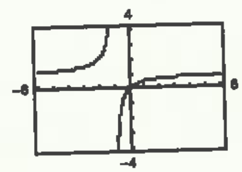
Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 0), (0, 1)$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

26. $f(x) = \dfrac{x}{x + 1}$



$f'(x) = \dfrac{(x + 1)(1) - (x)(1)}{(x + 1)^2} = \dfrac{1}{(x + 1)^2}$

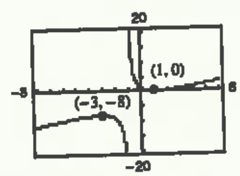
Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema

27. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$



$f'(x) = \dfrac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \dfrac{x^2 + 2x - 3}{(x + 1)^2} = \dfrac{(x + 3)(x - 1)}{(x + 1)^2}$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

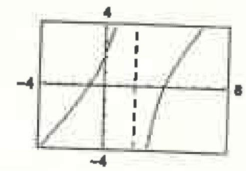
Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

28. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$



$$f'(x) = \frac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2} = \frac{x^2 - 4x + 10}{(x-2)^2}$$

Discontinuity: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema

29. $f(x) = \dfrac{x}{2} + \cos x, 0 < x < 2\pi$

$f'(x) = \dfrac{1}{2} - \sin x = 0$

Critical numbers: $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

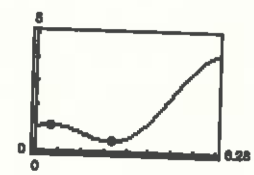
Test intervals:	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{6}, \dfrac{5\pi}{6}\right)$

Relative maximum: $\left(\dfrac{\pi}{6}, \dfrac{\pi + 6\sqrt{3}}{12}\right)$

Relative minimum: $\left(\dfrac{5\pi}{6}, \dfrac{5\pi - 6\sqrt{3}}{12}\right)$



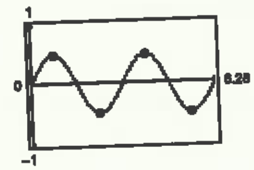
30. $f(x) = \sin x \cos x = \frac{1}{2}\sin 2x, 0 < x < 2\pi$

$f'(x) = \cos 2x = 0$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$



Decreasing on: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

Relative maxima: $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{5\pi}{4}, \frac{1}{2}\right)$

Relative minima: $\left(\frac{3\pi}{4}, -\frac{1}{2}\right), \left(\frac{7\pi}{4}, -\frac{1}{2}\right)$

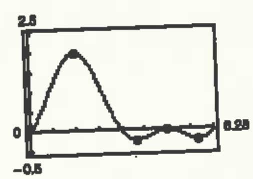
31. $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

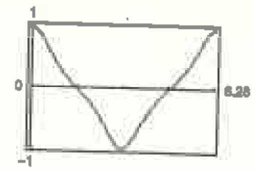


Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

32. $f(x) = \dfrac{\cos x}{1 + \sin^2 x}, 0 < x < 2\pi$



$$f'(x) = \frac{(1 + \sin^2 x)(-\sin x) - \cos x(2 \sin x \cos x)}{(1 + \sin^2 x)^2} = \frac{-\sin x(2 + \cos^2 x)}{(1 + \sin^2 x)^2} = 0$$

Critical number: $x = \pi$

Test intervals:	$0 < x < \pi$	$\pi < x < 2\pi$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(\pi, 2\pi)$

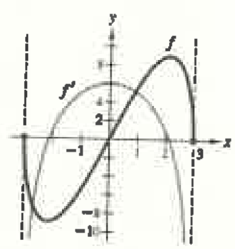
Decreasing on: $(0, \pi)$

Relative minimum: $(\pi, -1)$

33. $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

(a) $f'(x) = \dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)



(c) $\dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}} = 0$

Critical numbers: $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

(d) Intervals:

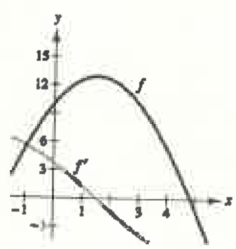
$\left(-3, -\frac{3\sqrt{2}}{2}\right)$	$\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$	$\left(\frac{3\sqrt{2}}{2}, 3\right)$
$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

34. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), [0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)



(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

$\left(0, \frac{3}{2}\right)$	$\left(\frac{3}{2}, 5\right)$
$f'(x) > 0$	$f'(x) < 0$
Increasing	Decreasing

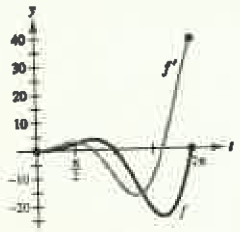
f is increasing when f' is positive and decreasing when f' is negative.

35. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t$

$= t(t \cos t + 2 \sin t)$

(b)



(c) $t(t \cos t + 2 \sin t) = 0$

$t = 0$ or $t = -2 \tan t$

$t \cot t = -2$

$t \approx 2.2889, 5.0870$ (graphing utility)

Critical numbers: $t = 2.2889, t = 5.0870$

(d) Intervals:

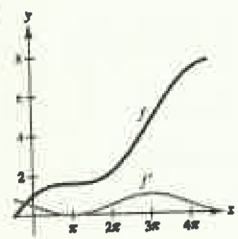
$(0, 2.2889)$	$(2.2889, 5.0870)$	$(5.0870, 2\pi)$
$f'(t) > 0$	$f'(t) < 0$	$f'(t) > 0$
Increasing	Decreasing	Increasing

f is increasing when f' is positive and decreasing when f' is negative.

36. $f(x) = \frac{x}{2} + \cos\frac{x}{2}, [0, 4\pi]$

(a) $f'(x) = \frac{1}{2} - \frac{1}{2}\sin\frac{x}{2}$

(b)



(c) $\frac{1}{2} - \frac{1}{2}\sin\frac{x}{2} = 0$

$\sin\frac{x}{2} = 1$

$\frac{x}{2} = \frac{\pi}{2}$

Critical number: $x = \pi$

(d) Intervals:

$(0, \pi)$	$(\pi, 4\pi)$
$f'(x) > 0$	$f'(x) > 0$
Increasing	Increasing

f is increasing when f' is positive.

In Exercises 37–42, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

37. $g(x) = f(x) + 5$

$g'(x) = f'(x)$

$g'(0) = f'(0) < 0$

38. $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

39. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(-6) = -f'(-6) < 0$

40. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(0) = -f'(0) > 0$

41. $g(x) = f(x - 10)$

$g'(x) = f'(x - 10)$

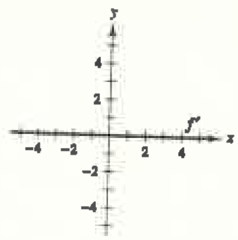
$g'(0) = f'(-10) > 0$

42. $g(x) = f(x - 10)$

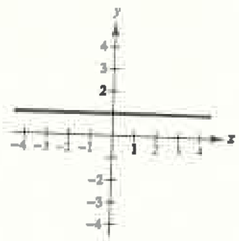
$g'(x) = f'(x - 10)$

$g'(8) = f'(-2) < 0$

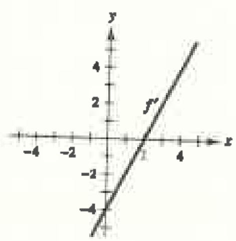
43. $f(x) = c$ is constant $\Rightarrow f'(x) = 0$



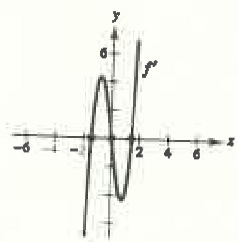
44. $f(x)$ is a line of slope $\approx 1 \Rightarrow f'(x) = 1$.



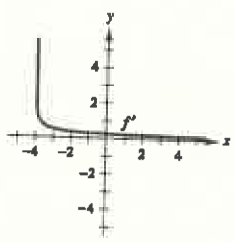
45. f is quadratic $\Rightarrow f'$ is a line.



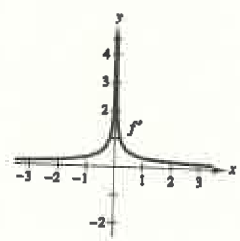
46. f is a 4^{th} degree polynomial $\Rightarrow f'$ is a cubic polynomial.



47. f has positive, but decreasing slope



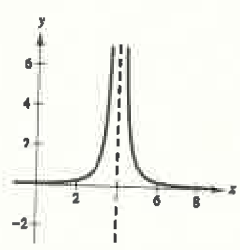
48. f has positive slope



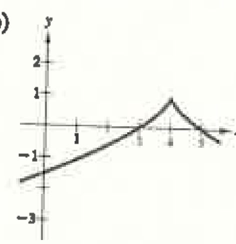
49. $f'(x) = \begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$

Two possibilities for $f(x)$ are given below.

(a)



(b)



50. Critical number: $x = 5$

$f'(4) = -2.5 \Rightarrow f$ is decreasing at $x = 4$.

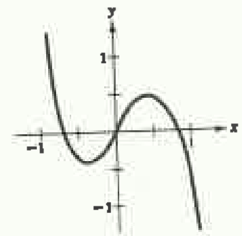
$f'(6) = 3 \Rightarrow f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

51. The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ since the sign of f' changes in these intervals. f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

Relative minimum when $x \approx -0.40$.

Relative maximum when $x \approx 0.48$.



52. $s(t) = 4.9(\sin\theta)t^2$

(a) $v(t) = 9.8(\sin\theta)t$ speed $= |9.8(\sin\theta)t|$

(b) If $\theta = \pi/2$, the speed is maximum,

$v(t) = 9.8t$.

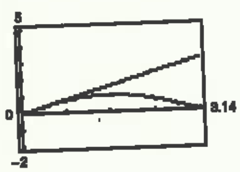
53. $f(x) = x$, $g(x) = \sin x$, $0 < x < \pi$

(a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.

(b)



$x > \sin x$ on $(0, \pi)$

(c) Let $h(x) = f(x) - g(x) = x - \sin x$

$h'(x) = 1 - \cos x > 0$ on $(0, \pi)$.

Therefore, $h(x)$ is increasing on $(0, \pi)$. Since $h(0) = 0$, $h(x) > 0$ on $(0, \pi)$. Thus,

$x - \sin x > 0$

$x > \sin x$

$f(x) > g(x)$ on $(0, \pi)$.

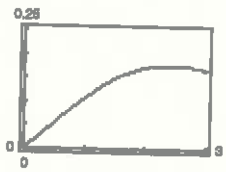
54. $C = \dfrac{3t}{27 + t^3}, t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greater near $t = 2.5$ hours.

(b)



The concentration is greatest when $t \approx 2.38$ hours.

(c) $C' = \dfrac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$

$= \dfrac{3(27 - 2t^3)}{(27 + t^3)^2}$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

55. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$v' = k(2Rr - 3r^2)$

$= kr(2R - 3r) = 0$

$r = 0$ or $\frac{2}{3}R$

Maximum when $r = \frac{2}{3}R$.

56. $P = 2.44x - \dfrac{x^2}{20{,}000} - 5000, 0 \leq x \leq 35{,}000$

$P' = 2.44 - \dfrac{x}{10{,}000} = 0$

$x = 24{,}400$

Test intervals:	$0 < x < 24{,}400$	$24{,}400 < x < 35{,}000$
Sign of P':	$P' > 0$	$P' < 0$

Increasing when $0 < x < 24{,}400$ hamburgers.

Decreasing when $24{,}400 < x < 35{,}000$ hamburgers.

57. $P = \dfrac{vR_1R_2}{(R_1 + R_2)^2}$, v and R_1 are constant

$\dfrac{dP}{dR_2} = \dfrac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$

$= \dfrac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Longrightarrow R_2 = R_1$

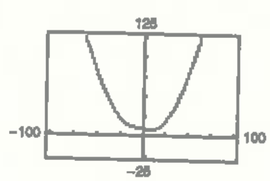
Maximum when $R_1 = R_2$.

58. $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

$T = 10°, R \approx 8.3666\Omega$

(b)

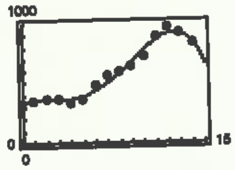


The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10°$.

59. (a) Using a graphing utility, you obtain

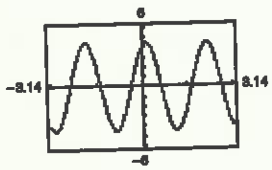
$$B = -0.1538t^4 + 3.5379t^3 - 19.7537t^2 + 42.3906t + 352.8234.$$

(b)



(c) The maximum of the model is $B = 951.5$ at $t = 12.58$.

60. $f(x) = 2\sin(3x) + 4\cos(3x)$



The maximum value is approximately 4.472. You could use calculus by finding $f'(x)$ and then observing that the maximum value of f occurs at a point where $f'(x) = 0$. For instance, $f'(0.154) \approx 0$, and $f(0.154) = 4.472$.

61. (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

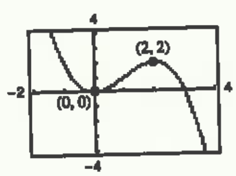
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$.

(0, 0):	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
(2, 2):	$2 = 8a_3 + 4a_2$	$(f(2) = 2)$
	$0 = 12a_3 + 4a_2$	$(f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0$, $a_2 = \frac{3}{2}$, $a_3 = -\frac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$

(d)



62. (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

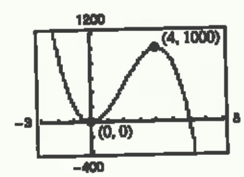
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

(0, 0):	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
(4, 1000):	$1000 = 64a_3 + 16a_2$	$(f(4) = 1000)$
	$0 = 48a_3 + 8a_2$	$(f'(4) = 0)$

(c) The solution is $a_0 = a_1 = 0$, $a_2 = \frac{375}{2}$, $a_3 = \frac{-125}{4}$

$$f(x) = \frac{-125}{4}x^3 + \frac{375}{2}x^2.$$

(d)



63. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

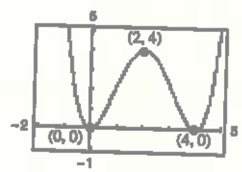
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(0, 0)$:	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
$(4, 0)$:	$0 = 256a_4 + 64a_3 + 16a_2$	$(f(4) = 0)$
	$0 = 256a_4 + 48a_3 + 8a_2$	$(f'(4) = 0)$
$(2, 4)$:	$4 = 16a_4 + 8a_3 + 4a_2$	$(f(2) = 4)$
	$0 = 32a_4 + 12a_3 + 4a_2$	$(f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0, a_2 = 4, a_3 = -2, a_4 = \frac{1}{4}$.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



64. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

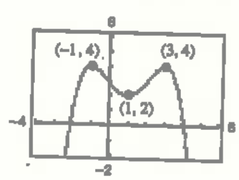
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(1, 2)$:	$2 = a_4 + a_3 + a_2 + a_1 + a_0$	$(f(1) = 2)$
	$0 = 4a_4 + 3a_3 + 2a_2 + a_1$	$(f'(1) = 0)$
$(-1, 4)$:	$4 = a_4 - a_3 + a_2 - a_1 + a_0$	$(f(-1) = 4)$
	$0 = -4a_4 + 3a_3 - 2a_2 + a_1$	$(f'(-1) = 0)$
$(3, 4)$:	$4 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$	$(f(3) = 4)$
	$0 = 108a_4 + 27a_3 + 6a_2 + a_1$	$(f'(3) = 0)$

(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}.$$

(d)



65. True

Let $h(x) = f(x) + g(x)$ where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ since $f'(x) > 0$ and $g'(x) > 0$.

66. False

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

67. False

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

68. True

If $f(x)$ is an nth-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

69. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, f is decreasing on the interval.

70. Suppose $f'(x)$ changes from positive to negative at c. Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b). By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b). Therefore, $f(c)$ is a maximum of f on (a, b) and thus, a relative maximum of f.

71. Let $f(x) = (1 + x)^n - nx - 1$. Then

$$\begin{aligned} f'(x) &= n(1 + x)^{n-1} - n \\ &= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1. \end{aligned}$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Longrightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Longrightarrow (1 + x)^n > 1 + nx.$$

Section 3.4 Concavity and the Second Derivative Test

1. $y = x^2 - x - 2, y'' = 2$

Concave upward: $(-\infty, \infty)$

2. $y = -x^3 + 3x^2 - 2, y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

3. $f(x) = \dfrac{24}{x^2 + 12}, y'' = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

4. $f(x) = \dfrac{x^2 - 1}{2x + 1}, y'' = \dfrac{-6}{(2x + 1)^3}$

Concave upward: $\left(-\infty, -\frac{1}{2}\right)$

Concave downward: $\left(-\frac{1}{2}, \infty\right)$

5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

6. $y = \dfrac{1}{270}(-3x^5 + 40x^3 + 135x), y'' = \dfrac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

7. $f(x) = 6x - x^2$

$f'(x) = 6 - 2x$

$f''(x) = -2$

Critical number: $x = 3$

$f''(3) < 0$

Therefore, $(3, 9)$ is relative maximum.

8. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

9. $f(x) = (x - 5)^2$

$f'(x) = 2(x - 5)$

$f''(x) = 2$

Critical number: $x = 5$

$f''(5) > 0$

Therefore, $(5, 0)$ is a relative minimum.

10. $f(x) = -(x - 5)^2$

$f'(x) = -2(x - 5)$

$f''(x) = -2$

Critical number: $x = 5$

$f''(5) < 0$

Therefore, $(5, 0)$ is a relative maximum.

11. $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

12. $f(x) = 5 + 3x^2 - x^3$

$f'(x) = 6x - 3x^2 = 3x(2 - x)$

$f''(x) = 6 - 6x = 6(1 - x)$

Critical numbers: $x = 0, x = 2$

$f''(0) = 6 > 0$

Therefore, $(0, 5)$ is a relative minimum.

$f''(2) = -6 < 0$

Therefore, $(2, 9)$ is a relative maximum.

13. $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

14. $f(x) = x^3 - 9x^2 + 27x$

$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$

$f''(x) = 6(x - 3)$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \geq 0$ for all x and, therefore, there are no relative extrema.

15. $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

16. $f(x) = \sqrt{x^2 + 1}$

$$f'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Critical number: $x = 0$

$$f''(x) = \frac{1}{(x^2 + 1)^{3/2}}$$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

17. $f(x) = x + \dfrac{4}{x}$

$f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{x^2 - 4}{x^2}$

$f''(x) = \dfrac{8}{x^3}$

Critical numbers: $x = \pm 2$

$f''(-2) < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) > 0$

Therefore, $(2, 4)$ is a relative minimum.

18. $f(x) = \dfrac{x}{x - 1}$

$f'(x) = \dfrac{-1}{(x - 1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

19. $f(x) = \cos x - x,\ 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, f is non-increasing and there are no relative extrema.

20. $f(x) = 2\sin x + \cos 2x,\ 0 \le x \le 2\pi$

$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x = 2\cos x(1 - 2\sin x) = 0$ when $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

$f''(x) = -2\sin x - 4\cos 2x$

$f''\left(\dfrac{\pi}{6}\right) < 0$

$f''\left(\dfrac{\pi}{2}\right) > 0$

$f''\left(\dfrac{5\pi}{6}\right) < 0$

$f''\left(\dfrac{3\pi}{2}\right) > 0$

Relative maxima: $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right), \left(\dfrac{5\pi}{6}, \dfrac{3}{2}\right)$

Relative minima: $\left(\dfrac{\pi}{2}, 1\right), \left(\dfrac{3\pi}{2}, -3\right)$

21. $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12 = 3(x + 2)(x - 2) = 0$ when $x = \pm 2$.

$f''(x) = 6x$

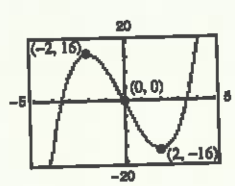
$f''(-2) = -12 < 0 \Rightarrow (-2, 16)$ is a relative maximum.

$f''(2) = 12 > 0 \Rightarrow (2, -16)$ is a relative minimum.

$f''(x) = 6x = 0$ when $x = 0$.

Test interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $(0, 0)$

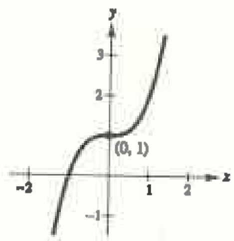


22. $f(x) = x^3 + 1$

$f'(x) = 3x^2 = 0$ when $x = 0$.

$f''(x) = 6x = 0$ when $x = 0$.

Since $f'(x) \geq 0$ for all x and the concavity changes at $x = 0$. $(0, 1)$ is a point of inflection.



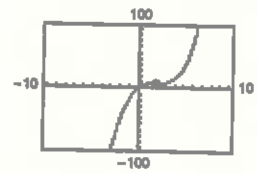
23. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$= 3(x - 2)^2 = 0$ when $x = 2$.

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

Since $f'(x) > 0$ when $x \neq 2$ and the concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.



24. $f(x) = 2x^3 - 3x^2 - 12x$

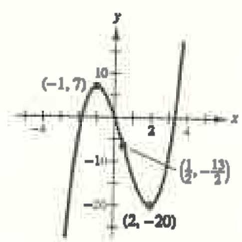
$f'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1) = 0$ when $x = -1, 2$.

$f''(x) = 12x - 6$

$f''(-1) = -18 < 0 \Longrightarrow (-1, 7)$ is a relative maximum.

$f''(2) = 18 > 0 \Longrightarrow (2, -20)$ is a relative minimum.

$f''(x) = 12x - 6 = 0$ when $x = \frac{1}{2}$.



Test interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{2}\right)$

25. $f(x) = \dfrac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x = x(x + 2)(x - 2) = 0$ when $x = 0, \pm 2$.

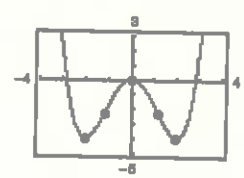
$f''(x) = 3x^2 - 4$

$f''(-2) = 8 > 0 \Longrightarrow (-2, -4)$ is a relative minimum.

$f''(0) = -4 < 0 \Longrightarrow (0, 0)$ is a relative maximum.

$f''(2) = 8 > 0 \Longrightarrow (2, -4)$ is a relative minimum.

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm\dfrac{2}{\sqrt{3}}$.



Test interval	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

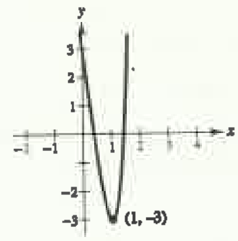
Points of inflection: $\left(\pm\dfrac{2}{\sqrt{3}}, -\dfrac{20}{9}\right)$

26. $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8 = 8(x - 1)(x^2 + x + 1) = 0$ when $x = 1$.

$f''(x) = 24x^2 = 0$ when $x = 0$.

Since $f''(1) > 0$, $(1, -3)$ is a relative minimum. However, $(0, 3)$ is not a point of inflection since $f''(x) \geq 0$ for all x.



27. $f(x) = x(x - 4)^3$

$f'(x) = x[3(x - 4)^2] + (x - 4)^3$

$= (x - 4)^2(4x - 4) = 4(x - 1)(x - 4)^2 = 0$ when $x = 1, 4$.

$f''(x) = 4(x - 1)[2(x - 4)] + 4(x - 4)^2$

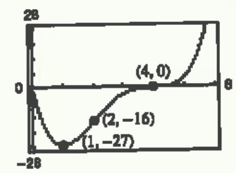
$= 4(x - 4)[2(x - 1) + (x - 4)]$

$= 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$

$f''(1) = 36 > 0 \Longrightarrow (1, -27)$ is a relative minimum.

$f''(4) = 0 \Longrightarrow$ test fails

By the First Derivative Test we see that $x = 4$ does not yield a relative extrema.



$f''(x) = 12(x - 4)(x - 2) = 0$ when $x = 2, 4$.

Test interval	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16)$, $(4, 0)$

28. $f(x) = x^3(x - 4)$

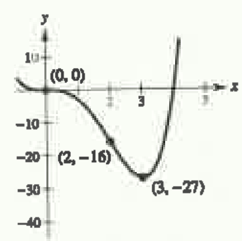
$f'(x) = x^3 + 3x^2(x - 4)$

$= x^2[x + 3(x - 4)] = 4x^2(x - 3) = 0$ when $x = 0, 3$.

$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$

$f''(0) = 0 \Longrightarrow$ test fails

$f''(3) = 36 > 0 \Longrightarrow (3, -27)$ is a relative minimum.



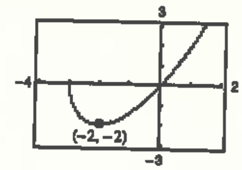
By the First Derivative Test we see that $x = 0$ does not yield a relative extrema.

$f''(x) = 12x(x - 2) = 0$ when $x = 0, 2$.

Test interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0)$, $(2, -16)$

29. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

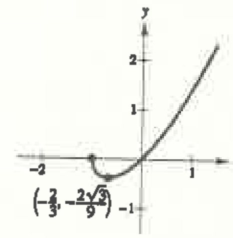


$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}} = 0 \text{ when } x = -2.$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

Since $f''(-2) > 0$, then $(-2, -2)$ is a relative minimum. $f''(x) > 0$ on the entire domain of f (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection.

30. $f(x) = x\sqrt{x+1}$, Domain: $[-1, \infty)$



$$f'(x) = (x)\frac{1}{2}(x+1)^{-1/2} + \sqrt{x+1} = \frac{3x+2}{2\sqrt{x+1}} = 0 \text{ when } x = -\frac{2}{3}.$$

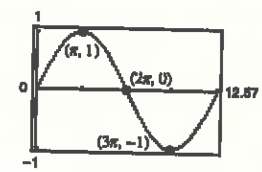
$$f''(x) = \frac{6\sqrt{x+1} - (3x+2)(x+1)^{-1/2}}{4(x+1)} = \frac{3x+4}{4(x+1)^{3/2}}$$

Since $f'\left(\frac{2}{3}\right) > 0$, then

$$\left(-\frac{2}{3}, -\frac{2\sqrt{3}}{9}\right)$$

is a relative minimum. $f''(x) > 0$ on the entire domain of f (except for $x = -1$, for which $f''(x)$ is undefined). There are no points of inflection.

31. $f(x) = \sin\left(\frac{x}{2}\right), 0 \le x \le 4\pi$



$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right) = 0$ when $x = \pi, 3\pi$.

$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$

$f''(\pi) < 0 \Longrightarrow (\pi, 1)$ is a relative maximum.

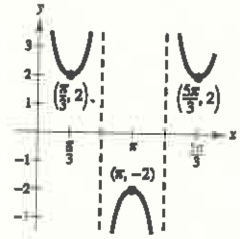
$f''(3\pi) > 0 \Longrightarrow (3\pi, -1)$ is a relative minimum.

$f''(x) = 0$ when $x = 0, 2\pi, 4\pi$.

Test interval	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $(2\pi, 0)$

32. $f(x) = 2\csc\frac{3x}{2}, 0 < x < 2\pi$



$f'(x) = -3\csc\frac{3x}{2}\cot\frac{3x}{2} = 0$ when $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

$f''(x) = \frac{9}{2}\left(\csc^3\frac{3x}{2} + \csc\frac{3x}{2}\cot^2\frac{3x}{2}\right) \neq 0$ for any x in the domain of f.

$f''\left(\frac{\pi}{3}\right) > 0 \Longrightarrow \left(\frac{\pi}{3}, 2\right)$ is a relative minimum.

$f''(\pi) < 0 \Longrightarrow (\pi, -2)$ is a relative maximum.

$f''\left(\frac{5\pi}{3}\right) > 0 \Longrightarrow \left(\frac{5\pi}{3}, 2\right)$ is a relative minimum.

33. $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$f'(x) = \sec\left(x - \frac{\pi}{2}\right)\tan\left(x - \frac{\pi}{2}\right) = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

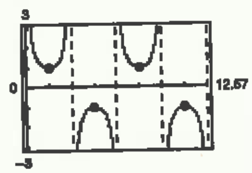
$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right)\tan^2\left(x - \frac{\pi}{2}\right) \neq 0$ for any x in the domain of f.

$f''\left(\frac{\pi}{2}\right) > 0 \Rightarrow \left(\frac{\pi}{2}, 1\right)$ is a relative minimum.

$f''\left(\frac{3\pi}{2}\right) < 0 \Rightarrow \left(\frac{3\pi}{2}, -1\right)$ is a relative maximum.

$f''\left(\frac{5\pi}{2}\right) > 0 \Rightarrow \left(\frac{5\pi}{2}, 1\right)$ is a relative minimum.

$f''\left(\frac{7\pi}{2}\right) < 0 \Rightarrow \left(\frac{7\pi}{2}, -1\right)$ is a relative maximum.



34. $f(x) = \sin x + \cos x, 0 \leq x \leq 2\pi$

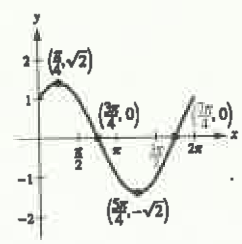
$f'(x) = \cos x - \sin x = 0$ when $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$f''(x) = -\sin x - \cos x$

$f''\left(\frac{\pi}{4}\right) < 0 \Rightarrow \left(\frac{\pi}{4}, \sqrt{2}\right)$ is a relative maximum.

$f''\left(\frac{5\pi}{4}\right) > 0 \Rightarrow \left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is a relative minimum.

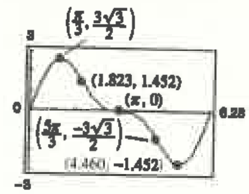
$f''(x) = 0$ when $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.



Test interval	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

35. $f(x) = 2\sin x + \sin 2x, 0 \le x \le 2\pi$



$f'(x) = 2\cos x + 2\cos 2x = 0$

$2(2\cos^2 x + \cos x - 1) = 0$

$(\cos x + 1)(2\cos x - 1) = 0$ when $\cos x = \frac{1}{2}, -1$ or $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

$f''(x) = -2\sin x - 4\sin 2x = -2\sin x(1 + 4\cos x)$

$f''\left(\frac{\pi}{3}\right) < 0 \Rightarrow \left(\frac{\pi}{3}, 2.598\right)$ is a relative maximum.

$f''\left(\frac{5\pi}{3}\right) > 0 \Rightarrow \left(\frac{5\pi}{3}, -2.598\right)$ is a relative minimum.

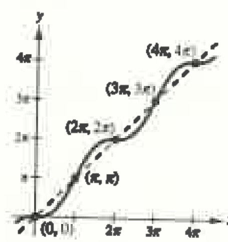
$f''(\pi) = 0$. By the first derivative test, $(\pi, 0)$ is not a relative extrema.

$f''(x) = 0$ when $x = 0, 1.823, \pi, 4.460$.

Test interval	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

36. $f(x) = x - \sin x, 0 \le x \le 4\pi$



$f'(x) = 1 - \cos x \ge 0$

Therefore, f is nondecreasing, and there are no relative extrema.

$f''(x) = \sin x = 0$ when $x = 0, \pi, 2\pi, 3\pi, 4\pi$.

Test interval	$0 < x < \pi$	$\pi < x < 2\pi$	$2\pi < x < 3\pi$	$3\pi < x < 4\pi$
Sign of $f''(x)$	$f'' > 0$	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward	Concave upward	Concave downward

Points of inflection: $(\pi, \pi), (2\pi, 2\pi), (3\pi, 3\pi)$

Note: $(0, 0)$ and $(4\pi, 4\pi)$ are not points of inflection since they are endpoints.

37. $f(x) = 0.2x^2(x - 3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x - 3)(10x^2 - 24x + 9)$

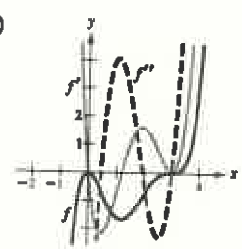
(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$f''(\frac{6}{5}) > 0 \Rightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

38. $f(x) = x^2\sqrt{6 - x^2}, [-\sqrt{6}, \sqrt{6}]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

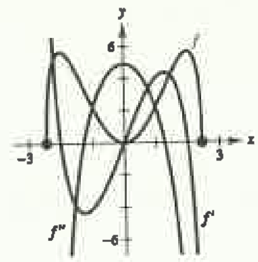
$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Longrightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Longrightarrow (\pm 2, 4\sqrt{2})$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

39. $f(x) = \sin x - \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x, [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{\pi}{2}, x = \dfrac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \dfrac{\pi}{6}, x = \dfrac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$

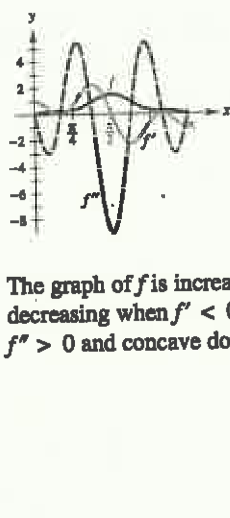
(b) $f''\left(\dfrac{\pi}{2}\right) < 0 \Longrightarrow \left(\dfrac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\dfrac{\pi}{6}, 0.2667\right)$, $(1.1731, 0.9638)$,

$(1.9685, 0.9637)$, $\left(\dfrac{5\pi}{6}, 0.2667\right)$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection since they are endpoints.

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

40. $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \dfrac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$$f''(x) = -\sqrt{2x}\sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}}$$

$$= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

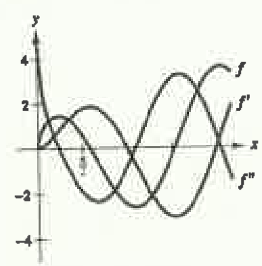
$$= \frac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

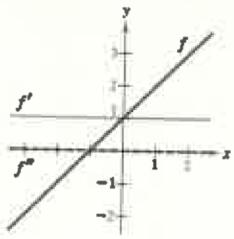
Points of inflection: $(0.75, 0.83)$, $(3.42, -0.72)$

(c)

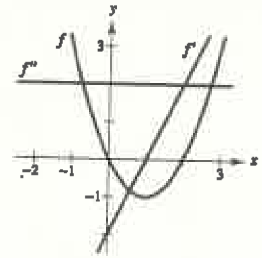


f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

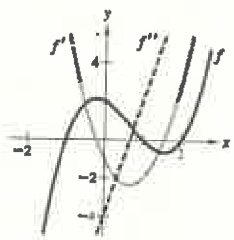
41.



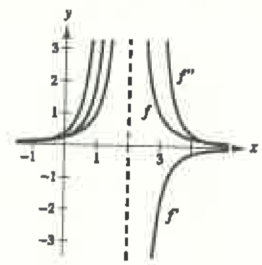
42.



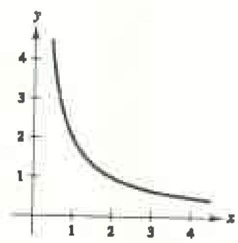
43.



44.



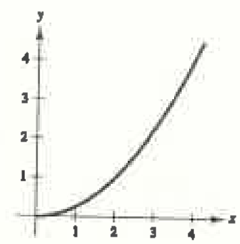
45. (a)



$f' < 0$ means f decreasing

f' increasing means concave upward

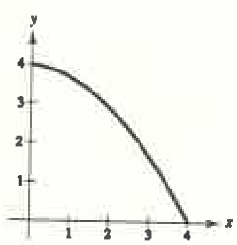
(b)



$f' > 0$ means f increasing

f' increasing means concave upward

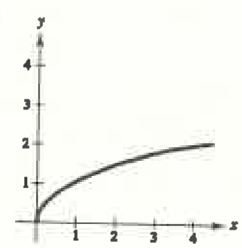
46. (a)



$f' < 0$ means f decreasing

f' decreasing means concave downwards

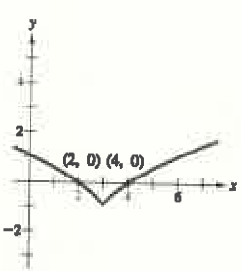
(b)



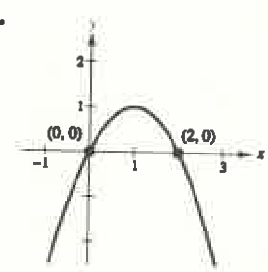
$f' > 0$ means f increasing

f' decreasing means concave downward

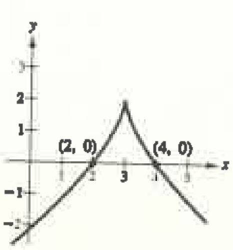
47.



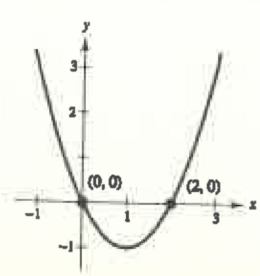
48.



49.



50.



51. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(3, 3)$

Relative minimum: $(5, 1)$

Point of inflection: $(4, 2)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{array}{l} f(3) = 27a + 9b + 3c + d = 3 \\ f(5) = 125a + 25b + 5c + d = 1 \end{array}\right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0, f''(4) = 24a + 2b = 0$

$$\begin{array}{ll} 49a + 8b + c = -1 & 24a + 2b = 0 \\ \underline{27a + 6b + c = 0} & \underline{22a + 2b = -1} \\ 22a + 2b = -1 & 2a = 1 \end{array}$$

$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

52. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{array}{l} f(2) = 8a + 4b + 2c + d = 4 \\ f(4) = 64a + 16b + 4c + d = 2 \end{array}\right\} 56a + 12b + 2c = -2 \Rightarrow 28a + 6b + c = -1$$

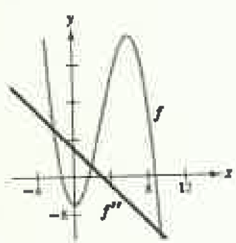
$f'(2) = 12a + 4b + c = 0,\ f'(4) = 48a + 8b + c = 0,\ f''(3) = 18a + 2b = 0$

$$\begin{array}{ll} 28a + 6b + c = -1 & 18a + 2b = 0 \\ \underline{12a + 4b + c = 0} & \underline{16a + 2b = -1} \\ 16a + 2b = -1 & 2a = 1 \end{array}$$

$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

53.



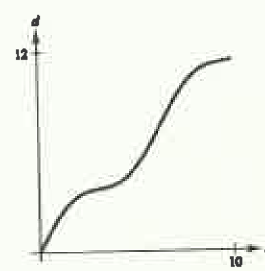
f'' is linear.

f' is quadratic.

f is cubic.

f concave upwards on $(-\infty, 3)$, downwards on $(3, \infty)$.

54. (a)



(b) Since the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

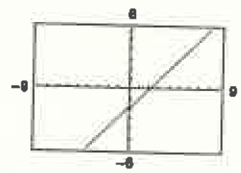
55. (a) $n = 1$:

$f(x) = x - c$

$f'(x) = 1$

$f''(x) = 0$

No inflection points



$n = 2$:

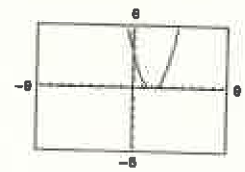
$f(x) = (x - 2)^2$

$f'(x) = 2(x - 2)$

$f''(x) = 2$

No inflection points

Relative minimum: $(2, 0)$



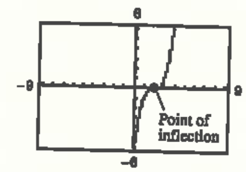
$n = 3$:

$f(x) = (x - 2)^3$

$f'(x) = 3(x - 2)^2$

$f''(x) = 6(x - 2)$

Inflection point: $(2, 0)$



$n = 4$:

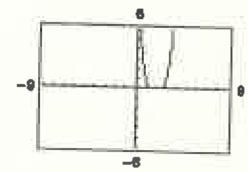
$f(x) = (x - 2)^4$

$f'(x) = 4(x - 2)^3$

$f''(x) = 12(x - 2)^2$

No inflection points:

Relative minimum: $(2, 0)$



Conclusion: If $n \geq 3$ and n is odd, then $(c, 0)$ is an inflection point. If $n \geq 2$ and n is even, then $(c, 0)$ is a relative minimum.

(b) Let $f(x) = (x - c)^n$, $f'(x) = n(x - c)^{n-1}$, $f''(x) = n(n - 1)(x - c)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = c$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = c$.

Thus, $x = c$ is an inflection point if and only if $n \geq 3$ is odd.

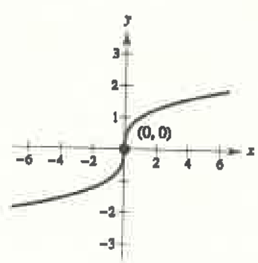
56. (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Inflection point: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



57. (a) The rate of change of sales is increasing.
$S'' > 0$

(b) The rate of change of sales is decreasing.
$S' > 0, S'' < 0$

(c) The rate of change of sales is constant.
$S' = C, S'' = 0$

(d) Sales are steady.
$S = C, S' = 0, S'' = 0$

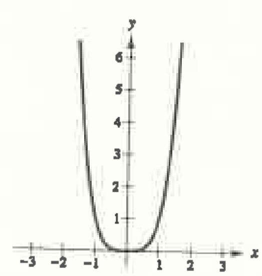
(e) Sales are declining, but at a lower rate.
$S' < 0, S'' > 0$

(f) Sale have bottomed out and have started to rise.
$S' > 0$

58. Let $f(x) = x^4$.

$f''(x) = 12x^2$

$f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



59. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \quad f''(x) = 6ax + 2b$

$f(0) = 0 \Rightarrow d = 0$

$$\begin{aligned}
f(-4) = 1 &\Rightarrow -64a + 16b - 4c = 1\\
f'(-4) = 0 &\Rightarrow \quad 48a - 8b + c = 0\\
f'(0) = 0 &\Rightarrow \quad c = 0
\end{aligned}$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$$

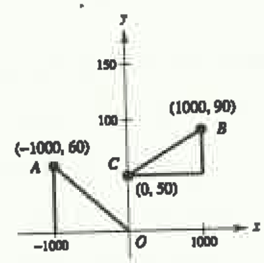
Two miles from touchdown.

60. (a) line OA: $y = -0.06x$ slope: -0.06

line CB: $y = 0.04x + 50$ slope: 0.04

$f(x) = ax^3 + bx^2 + cx + d$

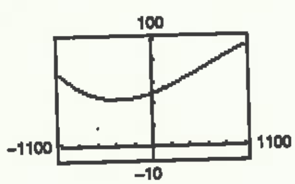
$f'(x) = 3ax^2 + 2bx + c$



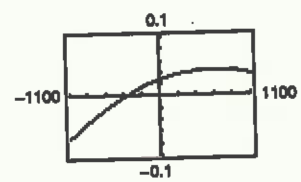
$$\begin{aligned}
(-1000, 60):&\quad 60 = (-1000)^3a + (1000)^2b - 1000c + d\\
&\quad -0.06 = (1000)^2 3a - 2000b + c\\
(1000, 90):&\quad 90 = (1000)^3a + (1000)^2b + 1000c + d\\
&\quad 0.04 = (1000)^2 3a + 2000b + c
\end{aligned}$$

The solution to this system of 4 equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



(c)



(d) The steepest part of the road is 6% at the point A.

61. $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

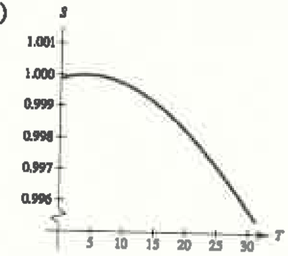
By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

62. $S = \dfrac{5.755T^3}{10^8} - \dfrac{8.521T^2}{10^6} + \dfrac{0.654T}{10^4} + 0.99987,\ \ 0 < T < 25$

(a) The maximum occurs when $T \approx 4°$ and $S \approx 0.999999$.

(b)



(c) $S(20°) \approx 0.9982$

63. $C = 0.5x^2 + 15x + 5000$

$\overline{C} = \dfrac{C}{x} = 0.5x + 15 + \dfrac{5000}{x}$

$\overline{C}$ = average cost per unit

$\dfrac{d\overline{C}}{dx} = 0.5 - \dfrac{5000}{x^2} = 0$ when $x = 100$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

64. $C = 2x + \dfrac{300{,}000}{x}$

$C' = 2 - \dfrac{300{,}000}{x^2} = 0$ when $x = 100\sqrt{15} \approx 387$

By the First Derivative Test, C is minimized when $x \approx 387$ units.

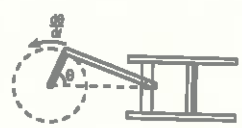
65. $v = -2400\pi \sin\theta$

$\dfrac{d\theta}{dt} = (200)(2\pi) = 400\pi$ rad/min

$\dfrac{dv}{dt} = -2400\pi\cos\theta \cdot \dfrac{d\theta}{dt}$

$= -2400\pi(400\pi)\cos\theta$

$= 0$ when $\theta = \dfrac{(2n+1)\pi}{2} = \dfrac{\pi}{2} + 2n\pi$, n any integer



By the First Derivative Test, v is maximum when $\theta = (3\pi/2) + 2n\pi$. The speed $|v|$ is maximum when $\theta = (\pi/2) + 2n\pi$ and $(3\pi/2) + 2n\pi$.

66. $E = kqx(x^2 + a^2)^{-3/2}$

$E' = kqx\left[-\dfrac{3}{2}(x^2 + a^2)^{-5/2}(2x)\right] + kq(x^2 + a^2)^{-3/2}$

$= kq(x^2 + a^2)^{-5/2}[-3x^2 + (x^2 + a^2)] = \dfrac{kq(a^2 - 2x^2)}{(x^2 + a^2)^{5/2}}$

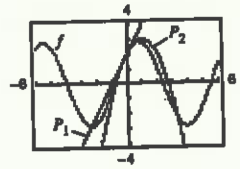
$E' = 0$ for $a^2 = 2x^2$ or $x = \ = \dfrac{a}{\sqrt{2}} = \dfrac{\sqrt{2}a}{2}$.

By the First Derivative Test, E is maximized when $x = \left(a\sqrt{2}\right)/2$.

67. $f(x) = 2(\sin x + \cos x)$, $\quad f(0) = 2$

$f'(x) = 2(\cos x - \sin x)$, $\quad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x)$, $\quad f''(0) = -2$



$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

$P_2(x) = 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$

$P_2'(x) = 2 - 2x$

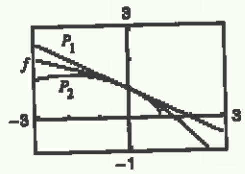
$P_2''(x) = -2$

The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

68. $f(x) = \sqrt{1 - x}$, $\quad f(0) = 1$

$f'(x) = -\dfrac{1}{2\sqrt{1 - x}}$, $\quad f'(0) = -\dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4(1 - x)^{3/2}}$, $\quad f''(0) = -\dfrac{1}{4}$



$P_1(x) = 1 + \left(-\dfrac{1}{2}\right)(x - 0) = 1 - \dfrac{x}{2}$

$P_1'(x) = -\dfrac{1}{2}$

$P_2(x) = 1 + \left(-\dfrac{1}{2}\right)(x - 0) + \dfrac{1}{2}\left(-\dfrac{1}{4}\right)(x - 0)^2 = 1 - \dfrac{x}{2} - \dfrac{x^2}{8}$

$P_2'(x) = -\dfrac{1}{2} - \dfrac{x}{4}$

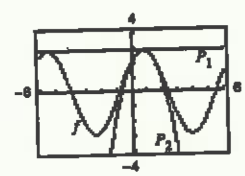
$P_2''(x) = -\dfrac{1}{4}$

The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

69. $f(x) = 2(\sin x + \cos x)$, $\quad f\left(\dfrac{\pi}{4}\right) = 2\sqrt{2}$

$f'(x) = 2(\cos x - \sin x)$, $\quad f'\left(\dfrac{\pi}{4}\right) = 0$

$f''(x) = 2(-\sin x - \cos x)$, $\quad f''\left(\dfrac{\pi}{4}\right) = -2\sqrt{2}$



$P_1(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) = 2\sqrt{2}$

$P_1'(x) = 0$

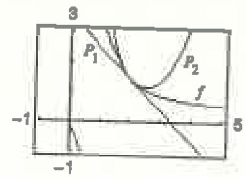
$P_2(x) = 2\sqrt{2} + 0\left(x - \dfrac{\pi}{4}\right) + \dfrac{1}{2}(-2\sqrt{2})\left(x - \dfrac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \dfrac{\pi}{4}\right)^2$

$P_2'(x) = -2\sqrt{2}\left(x - \dfrac{\pi}{4}\right)$

$P_2''(x) = -2\sqrt{2}$

The values of f, P_1, P_2, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

70. $f(x) = \dfrac{\sqrt{x}}{x-1}$, $\qquad f(2) = \sqrt{2}$



$f'(x) = \dfrac{-(x+1)}{2\sqrt{x}(x-1)^2}$, $\qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2+6x-1}{4x^{3/2}(x-1)^3}$, $\qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

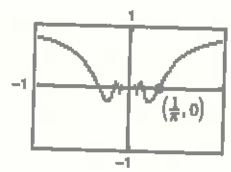
$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x-2) + \dfrac{23\sqrt{2}}{32}(x-2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x-2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$

The values of f, P_1, P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

71. $f(x) = x\sin\left(\dfrac{1}{x}\right)$



$f'(x) = x\left[-\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)\right] + \sin\left(\dfrac{1}{x}\right) = -\dfrac{1}{x}\cos\left(\dfrac{1}{x}\right) + \sin\left(\dfrac{1}{x}\right)$

$f''(x) = -\dfrac{1}{x}\left[\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)\right] + \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) - \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^3}\sin\left(\dfrac{1}{x}\right) = 0$

$x = \dfrac{1}{\pi}$

Point of inflection: $\left(\dfrac{1}{\pi}, 0\right)$

When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.

72. $f(x) = x(x-6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0$

$f''(x) = 6x - 24 = 6(x-4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of f.

73. Assume the zeros of f are all real. Then express the function as $f(x) = a(x - r_1)(x - r_2)(x - r_3)$ where r_1, r_2, and r_3 are the distinct zeros of f. From the Product Rule for a function involving three factors, we have

$$f'(x) = a[(x - r_1)(x - r_2) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)]$$
$$f''(x) = a[(x - r_1) + (x - r_2) + (x - r_1) + (x - r_3) + (x - r_2) + (x - r_3)]$$
$$= a[6x - 2(r_1 + r_2 + r_3)].$$

Consequently, $f''(x) = 0$ if

$$x = \frac{2(r_1 + r_2 + r_3)}{6} = \frac{r_1 + r_2 + r_3}{3} = (\text{Average of } r_1, r_2, \text{ and } r_3).$$

74.
$$p(x) = ax^3 + bx^2 + cx + d$$
$$p'(x) = 3ax^2 + 2bx + c$$
$$p''(x) = 6ax + 2b$$
$$6ax + 2b = 0$$
$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$
$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

75. Suppose $y_1 < d < y_2$ and let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$, and therefore, by the Extreme Value Theorem, attains a minimum value at some point c in $[a, b]$. This point c cannot be the endpoints a or b because

$$g'(a) = f'(a) - d = y_1 - d < 0$$
$$g'(b) = f'(b) - d = y_2 - d > 0.$$

Hence, g attains its minimum value at c, $a < c < b$. This implies that $g'(c) = 0$ and $f'(c) = d$. The proof for $y_2 < d < y_1$ is similar.

76. The First Derivative Test has the following advantages:

1. It does not fail as the Second Derivative Test sometimes does.

   Example: $f(x) = x^3$

2. It can be used for critical numbers that make the derivative undefined. The Second Derivative Test can only be used if $f'(c) = 0$.

   Example: $f(x) = (x - 1)^{2/3}$

3. You do not need to take a second derivative to determine relative extrema.

   Example: $f(x) = \dfrac{\sqrt{x + 7}}{x^2 - 9}$

The Second Derivative Test usually works best on polynomials or on functions whose derivatives are easily determined. It has the added advantage of providing additional information such as concavity and points of inflection.

77. True. Let $y = ax^3 + bx^2 + cx + d, a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

78. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

79. False

$$f(x) = 3\sin x + 2\cos x$$
$$f'(x) = 3\cos x - 2\sin x$$
$$3\cos x - 2\sin x = 0$$
$$3\cos x = 2\sin x$$
$$\tfrac{3}{2} = \tan x$$

Critical number: $x = \tan^{-1}\left(\frac{3}{2}\right)$

$f\left(\tan^{-1}\frac{3}{2}\right) \approx 3.60555$ is the maximum value of y.

80. True

$$y = \sin(bx)$$

Slope: $y' = b\cos(bx)$

$-b \le y' \le b$ (Assume $b > 0$)

Section 3.5 Limits at Infinity

1. $f(x) = \dfrac{3x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 3$

Matches (f)

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

3. $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

Matches (d)

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

5. $f(x) = \dfrac{4\sin x}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptotes: $y = 0$

Matches (b)

6. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

No vertical asymptotes

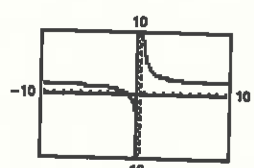
Horizontal asymptote: $y = 2$

Matches (e)

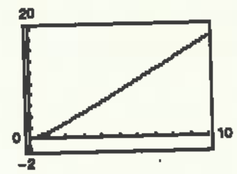
7. $f(x) = \dfrac{4x + 3}{2x - 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2

$\lim_{x\to\infty} f(x) = 2$



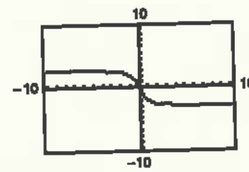
8. $f(x) = \dfrac{2x^2}{x+1}$



x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19,998	199,998	1,999,998

$\lim_{x\to\infty} f(x) = \infty$ (Limit does not exist.)

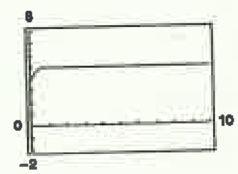
9. $f(x) = \dfrac{-6x}{\sqrt{4x^2+5}}$



x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-2	-2.98	-2.9998	-3	-3	-3	-3

$\lim_{x\to\infty} f(x) = -3$

10. $f(x) = 5 - \dfrac{1}{x^2+1}$



x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5

$\lim_{x\to\infty} f(x) = 5$

11. $\lim_{x\to\infty} \dfrac{2x-1}{3x+2} = \lim_{x\to\infty} \dfrac{2-(1/x)}{3+(2/x)} = \dfrac{2-0}{3+0} = \dfrac{2}{3}$

12. $\lim_{x\to\infty} \dfrac{5x^3+1}{10x^3-3x^2+7} = \lim_{x\to\infty} \dfrac{5+(1/x^3)}{10-(3/x)+(7/x^3)} = \dfrac{1}{2}$

13. $\lim_{x\to\infty} \dfrac{x}{x^2-1} = \lim_{x\to\infty} \dfrac{1/x}{1-(1/x^2)} = \dfrac{0}{1} = 0$

14. $\lim_{x\to\infty} \dfrac{2x^{10}-1}{10x^{11}-3} = \lim_{x\to\infty} \dfrac{(2/x)-(1/x^{11})}{10-(3/x^{11})} = \dfrac{0}{10} = 0$

15. $\lim_{x\to-\infty} \dfrac{5x^2}{x+3} = \lim_{x\to-\infty} \dfrac{5x}{1+(3/x)} = -\infty$

Limit does not exist.

16. $\lim_{x\to\infty} \left(2x - \dfrac{1}{x^2}\right) = \infty - 0 = \infty$

Limit does not exist.

17. $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}} = \lim_{x\to-\infty} \dfrac{1}{\dfrac{\sqrt{x^2-x}}{-\sqrt{x^2}}}$, $\left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

$$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1-(1/x)}} = -1$$

18. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1+(1/x^2)}} = 1$

19. $\lim_{x\to\infty} \dfrac{2x+1}{\sqrt{x^2-x}} = \lim_{x\to\infty} \dfrac{2+(1/x)}{\sqrt{1-(1/x)}} = 2$

20. $\lim_{x\to-\infty} \frac{-3x+1}{\sqrt{x^2+x}} = \lim_{x\to-\infty} \frac{-3+(1/x)}{\frac{\sqrt{x^2+x}}{-\sqrt{x^2}}}$, (for $x < 0$ we have $-\sqrt{x^2} = x$)

$$= \lim_{x\to-\infty} \frac{3-(1/x)}{\sqrt{1+(1/x)}} = 3$$

21. Since $(-1/x) \le (\sin(2x))/x \le (1/x)$ for all $x \ne 0$, we have by the Squeeze Theorem,

$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin(2x)}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin(2x)}{x} \le 0.$$

Therefore, $\lim_{x\to\infty} \frac{\sin(2x)}{x} = 0.$

22. $\lim_{x\to\infty} \frac{x-\cos x}{x} = \lim_{x\to\infty} \left(1 - \frac{\cos x}{x}\right)$

$$= 1 - 0 = 1$$

Note:

$\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by the Squeeze Theorem since

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}.$$

23. $\lim_{x\to\infty} \frac{1}{2x+\sin x} = 0$

24. $\lim_{x\to\infty} \sin\frac{1}{x} = \sin 0 = 0$

25. $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to0^+} \frac{\sin t}{t} = 1$

(Let $x = 1/t$.)

26. $\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{t\to0^+} \frac{\tan t}{t} = \lim_{t\to0^+} \left[\frac{\sin t}{t} \cdot \frac{1}{\cos t}\right]$

$$= (1)(1) = 1$$

(Let $x = 1/t$.)

27. $\lim_{x\to-\infty} \left(x+\sqrt{x^2+3}\right) = \lim_{x\to-\infty} \left[\left(x+\sqrt{x^2+3}\right) \cdot \frac{x-\sqrt{x^2+3}}{x-\sqrt{x^2+3}}\right] = \lim_{x\to-\infty} \frac{-3}{x-\sqrt{x^2+3}} = 0$

28. $\lim_{x\to\infty} \left(2x-\sqrt{4x^2+1}\right) = \lim_{x\to\infty} \left[\left(2x-\sqrt{4x^2+1}\right) \cdot \frac{2x+\sqrt{4x^2+1}}{2x+\sqrt{4x^2+1}}\right] = \lim_{x\to\infty} \frac{-1}{2x+\sqrt{4x^2+1}} = 0$

29. $\lim_{x\to\infty} \left(x-\sqrt{x^2+x}\right) = \lim_{x\to\infty} \left[\left(x-\sqrt{x^2+x}\right) \cdot \frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right]$

$$= \lim_{x\to\infty} \frac{-x}{x+\sqrt{x^2+x}} = \lim_{x\to\infty} \frac{-1}{1+\sqrt{1+(1/x)}} = -\frac{1}{2}$$

30. $\lim_{x\to-\infty} \left(3x+\sqrt{9x^2-x}\right) = \lim_{x\to-\infty} \left[\left(3x+\sqrt{9x^2-x}\right) \cdot \frac{3x-\sqrt{9x^2-x}}{3x-\sqrt{9x^2-x}}\right]$

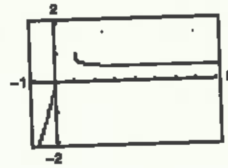
$$= \lim_{x\to-\infty} \frac{x}{3x-\sqrt{9x^2-x}}$$

$$= \lim_{x\to-\infty} \frac{1}{3-\frac{\sqrt{9x^2-x}}{-\sqrt{x^2}}} \quad \text{(for } x < 0 \text{ we have } x = -\sqrt{x^2}\text{)}$$

$$= \lim_{x\to-\infty} \frac{1}{3+\sqrt{9-(1/x)}} = \frac{1}{6}$$

31.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500



$$\lim_{x\to\infty}\left(x-\sqrt{x(x-1)}\right)=\lim_{x\to\infty}\frac{x-\sqrt{x^2-x}}{1}\cdot\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}}$$

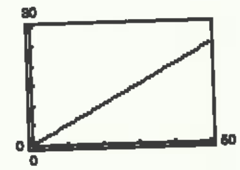
$$=\lim_{x\to\infty}\frac{x}{x+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{1}{1+\sqrt{1-(1/x)}}$$

$$=\frac{1}{2}$$

32.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

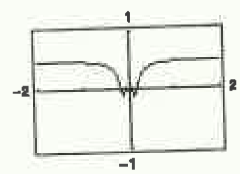


$$\lim_{x\to\infty}\frac{x^2-x\sqrt{x^2-x}}{1}\cdot\frac{x^2+x\sqrt{x^2-x}}{x^2+x\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{x^3}{x^2+x\sqrt{x^2-x}}=\infty$$

Limit does not exist.

33.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

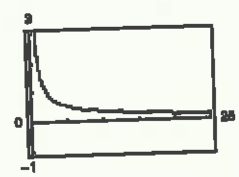


Let $x = 1/t$.

$$\lim_{x\to\infty} x\sin\left(\frac{1}{2x}\right)=\lim_{t\to 0^+}\frac{\sin(t/2)}{t}=\lim_{t\to 0^+}\frac{1}{2}\,\frac{\sin(t/2)}{t/2}=\frac{1}{2}$$

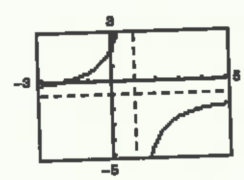
34.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001



$$\lim_{x\to\infty}\frac{x+1}{x\sqrt{x}}=0$$

35. $y=\dfrac{2+x}{1-x}$



Intercepts: $(-2, 0), (0, 2)$

Symmetry: none

Horizontal asymptote: $y=-1$ since

$$\lim_{x\to-\infty}\frac{2+x}{1-x}=-1=\lim_{x\to\infty}\frac{2+x}{1-x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)

36. $y = \dfrac{x-3}{x-2}$

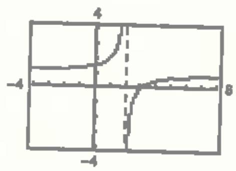
Intercepts: $(3, 0), \left(0, \dfrac{3}{2}\right)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x-3}{x-2} = 1 = \lim_{x \to \infty} \frac{x-3}{x-2}.$$

Discontinuity: $x = 2$ (Vertical asymptote)



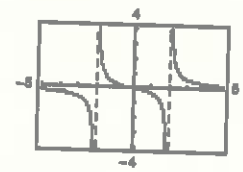
37. $y = \dfrac{x}{x^2 - 4}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$



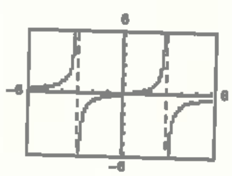
38. $y = \dfrac{2x}{9 - x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 3$



39. $y = \dfrac{x^2}{x^2 + 9}$

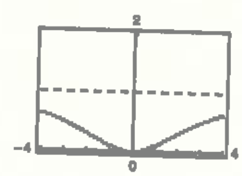
Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x^2}{x^2+9} = 1 = \lim_{x \to \infty} \frac{x^2}{x^2+9}.$$

Relative minimum: $(0, 0)$



40. $y = \dfrac{x^2}{x^2 - 9}$

Intercept: $(0, 0)$

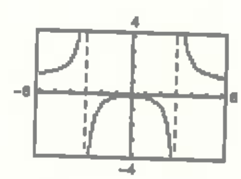
Symmetry: y-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x^2}{x^2-9} = 1 = \lim_{x \to \infty} \frac{x^2}{x^2-9}.$$

Discontinuities: $x = \pm 3$ (Vertical asymptotes)

Relative maximum: $(0, 0)$



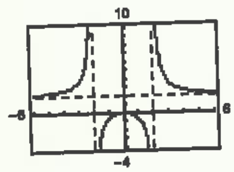
41. $y = \dfrac{2x^2}{x^2 - 4}$

Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Vertical asymptote: $x = \pm 2$



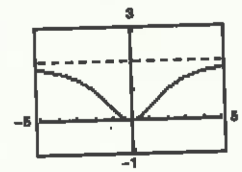
42. $y = \dfrac{2x^2}{x^2 + 4}$

Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Relative minimum: $(0, 0)$



43. $xy^2 = 4$

Domain: $x > 0$

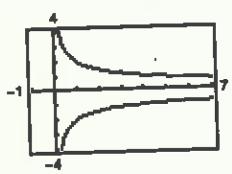
Intercepts: none

Symmetry: x-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \to \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



44. $x^2y = 4$

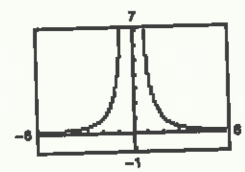
Intercepts: none

Symmetry: y-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{4}{x^2} = 0 = \lim_{x \to \infty} \frac{4}{x^2}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



45. $y = \dfrac{2x}{1 - x}$

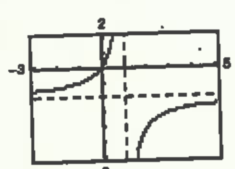
Intercept: $(0, 0)$

Symmetry: none

Horizontal asymptote: $y = -2$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \to \infty} \frac{2x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



46. $y = \dfrac{2x}{1 - x^2}$

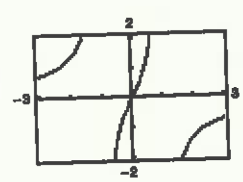
Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x^2} = 0 = \lim_{x \to \infty} \frac{2x}{1 - x^2}.$$

Discontinuities: $x = \pm 1$ (Vertical asymptotes)



47. $y = 2 - \dfrac{3}{x^2}$

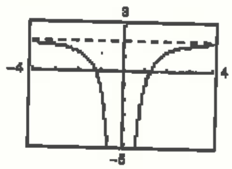
Intercepts: $\left(\pm\sqrt{3/2}, 0\right)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$ since

$$\lim_{x\to-\infty}\left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x\to\infty}\left(2 - \frac{3}{x^2}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



48. $y = 1 + \dfrac{1}{x}$

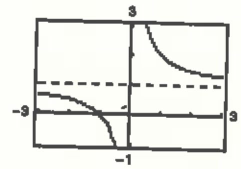
Intercept: $(-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x\to-\infty}\left(1 + \frac{1}{x}\right) = 1 = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



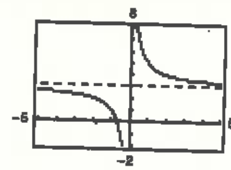
49. $y = 3 + \dfrac{2}{x}$

Intercept: $y = 0 = 3 + \dfrac{2}{x} \Rightarrow \dfrac{2}{x} = -3 \Rightarrow x = -\dfrac{2}{3}$ $\left(-\dfrac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$



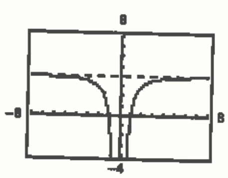
50. $y = 4\left(1 - \dfrac{1}{x^2}\right)$

Intercepts: $(\pm 1, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 0$



51. $y = \dfrac{x^3}{\sqrt{x^2 - 4}}$

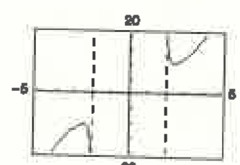
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)



52. $y = \dfrac{x}{\sqrt{x^2 - 4}}$

Domain: $(-\infty, -2), (2, \infty)$

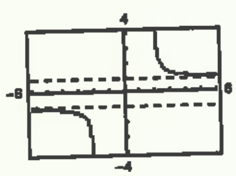
Intercepts: none

Symmetry: origin

Horizontal asymptotes: $y = \pm 1$ since

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2-4}} = 1, \lim_{x\to-\infty} \frac{x}{\sqrt{x^2-4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)



53. $f(x) = 5 - \dfrac{1}{x^2} = \dfrac{5x^2 - 1}{x^2}$

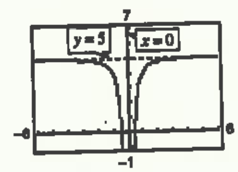
Domain: $(-\infty, 0), (0, \infty)$

$f'(x) = \dfrac{2}{x^3} \Rightarrow$ No relative extrema

$f''(x) = -\dfrac{6}{x^4} \Rightarrow$ No points of inflection

Vertical asymptote: $x = 0$

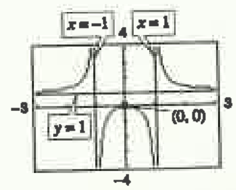
Horizontal asymptote: $y = 5$



54. $f(x) = \dfrac{x^2}{x^2 - 1}$

$$f'(x) = \frac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{(x^2-1)^2(-2) + 2x(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{2(3x^2+1)}{(x^2-1)^3}$$



Since $f''(0) < 0$, then $(0, 0)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$

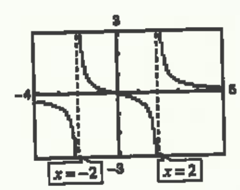
55. $f(x) = \dfrac{x}{x^2 - 4}$

$$f'(x) = \frac{(x^2-4) - x(2x)}{(x^2-4)^2}$$

$$= \frac{-(x^2+4)}{(x^2-4)^2} \neq 0 \text{ for any } x \text{ in the domain of } f.$$

$$f''(x) = \frac{(x^2-4)^2(-2x) + (x^2+4)(2)(x^2-4)(2x)}{(x^2-4)^4}$$

$$= \frac{2x(x^2+12)}{(x^2-4)^3} = 0 \text{ when } x = 0.$$



Since $f''(x) > 0$ on $(-2, 0)$ and $f''(x) < 0$ on $(0, 2)$, then $(0, 0)$ is a point of inflection.

Vertical asymptotes: $x = \pm 2$

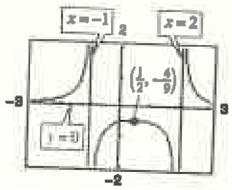
Horizontal asymptote: $y = 0$

56. $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x+1)(x-2)}$

$f'(x) = \dfrac{-(2x-1)}{(x^2 - x - 2)^2} = 0$ when $x = \dfrac{1}{2}$.

$f''(x) = \dfrac{(x^2 - x - 2)^2(-2) + (2x-1)(2)(x^2 - x - 2)(2x-1)}{(x^2 - x - 2)^4}$

$= \dfrac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}$



Since $f''\left(\frac{1}{2}\right) < 0$, then $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

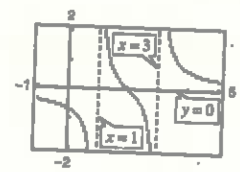
Horizontal asymptote: $y = 0$

57. $f(x) = \dfrac{x-2}{x^2 - 4x + 3} = \dfrac{x-2}{(x-1)(x-3)}$

$f'(x) = \dfrac{(x^2 - 4x + 3) - (x-2)(2x-4)}{(x^2 - 4x + 3)^2} = \dfrac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$

$f''(x) = \dfrac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4}$

$= \dfrac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = 0$ when $x = 2$.



Since $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptote: $x = 1, x = 3$

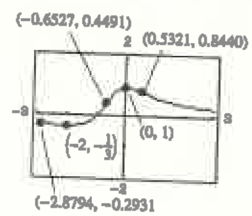
Horizontal asymptote: $y = 0$

58. $f(x) = \dfrac{x+1}{x^2 + x + 1}$

$f'(x) = \dfrac{-x(x+2)}{(x^2 + x + 1)^2} = 0$ when $x = 0, -2$.

$f''(x) = \dfrac{2(x^3 + 3x^2 - 1)}{(x^2 + x + 1)^3} = 0$ when $x \approx 0.5321, -0.6527, -2.8794$.

$f''(0) < 0$



Therefore, $(0, 1)$ is a relative maximum.

$f''(-2) > 0$

Therefore,

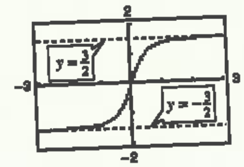
$\left(-2, -\dfrac{1}{3}\right)$

is a relative minimum.

Points of inflection: $(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and $(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$

59. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$



$f'(x) = \dfrac{3}{(4x^2 + 1)^{3/2}} \Rightarrow$ No relative extrema

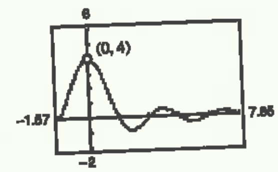
$f''(x) = \dfrac{-36x}{(4x^2 + 1)^{5/2}} = 0$ when $x = 0$.

Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm\dfrac{3}{2}$

No vertical asymptotes

60. $f(x) = \dfrac{2 \sin 2x}{x}$



$f'(x) = \dfrac{4x \cos 2x - 2 \sin 2x}{x^2}$

There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

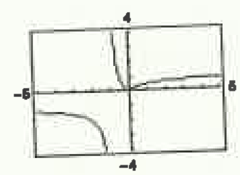
Relative minima: $(\pm 2.25, -0.869)$, $(\pm 5.45, -0.365)$

Relative maxima: $(\pm 3.87, 0.513)$

Horizontal asymptote: $y = 0$

No vertical asymptotes

61. (a) $f(x) = \dfrac{|x|}{x + 1}$

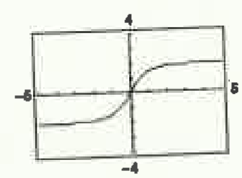


$\displaystyle\lim_{x \to \infty} \frac{|x|}{x + 1} = 1$

$\displaystyle\lim_{x \to -\infty} \frac{|x|}{x + 1} = -1$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.

(b) $f(x) = \dfrac{2x}{\sqrt{x^2 + 1}}$



$\displaystyle\lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + 1}} = 2$

$\displaystyle\lim_{x \to -\infty} \frac{2x}{\sqrt{x^2 + 1}} = -2$

Therefore, $y = 2$ and $y = -2$ are both horizontal asymptotes.

62. (a) $h(x) = \dfrac{f(x)}{x^2}$

$$= \frac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \frac{10}{x^2}$$

$$\lim_{x \to \infty} h(x) = \infty - 3 = \infty$$

Limit does not exist.

(b) $h(x) = \dfrac{f(x)}{x^3}$

$$= \frac{5x^3 - 3x^2 + 10}{x^3} = 5 - \frac{3}{x} + \frac{10}{x^3}$$

$$\lim_{x \to \infty} h(x) = 5$$

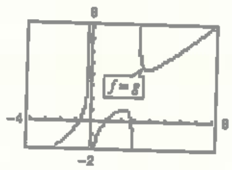
(c) $h(x) = \dfrac{f(x)}{x^4}$

$$= \frac{5x^3 - 3x^2 + 10}{x^4} = \frac{5}{x} - \frac{3}{x^2} + \frac{10}{x^4}$$

$$\lim_{x \to \infty} h(x) = 0$$

63. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$, $g(x) = x + \dfrac{2}{x(x - 3)}$

(a)

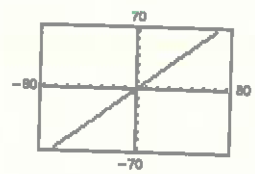


(b) $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x - 3)}$

$$= \frac{x^2(x - 3)}{x(x - 3)} + \frac{2}{x(x - 3)}$$

$$= x + \frac{2}{x(x - 3)} = g(x)$$

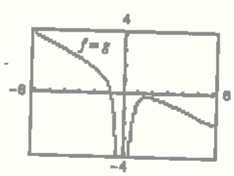
(c)



The graph appears as the slant asymptote $y = x$.

64. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$, $g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

(a)

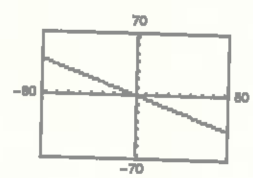


(b) $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$$= -\left[\frac{x^3}{2x^2} - \frac{2x^2}{2x^2} + \frac{2}{2x^2}\right]$$

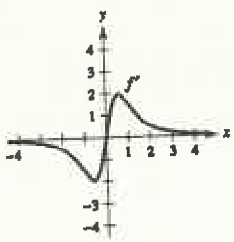
$$= -\frac{1}{2}x + 1 - \frac{1}{x^2} = g(x)$$

(c)



The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

65. (a)



(b) $\lim_{x \to \infty} f(x) = 3 \qquad \lim_{x \to \infty} f'(x) = 0$

(c) Since $\lim_{x \to \infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x \to \infty} f'(x) = 0$.

66. $\lim_{v_1/v_2 \to \infty} 100\left[1 - \dfrac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

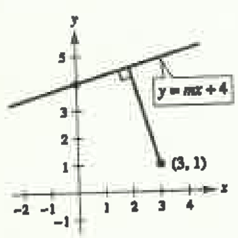
67. $C = 0.5x + 500$

$$\overline{C} = \frac{C}{x}$$

$$\overline{C} = 0.5 + \frac{500}{x}$$

$$\lim_{x \to \infty}\left(0.5 + \frac{500}{x}\right) = 0.5$$

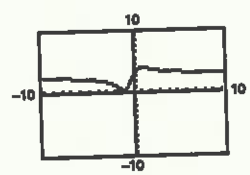
68. line: $mx - y + 4 = 0$



(a) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}}$

$$= \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$

(b)

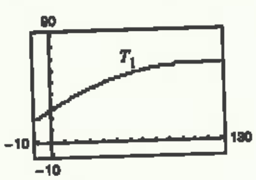


(c) $\lim_{m \to \infty} d(m) = 3 = \lim_{m \to -\infty} d(m)$

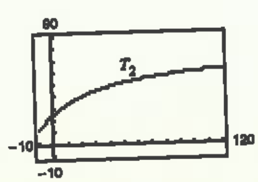
The line approaches the vertical line. Hence, the distance approaches 3, $x = 0$.

69. (a) $T_1(t) = -0.003t^2 + 0.677t + 26.564$

(b)



(c)



$$T_2 = \frac{2468 + 155t}{2(50 + t)}$$

(d) $T_1(0) \approx 26.6$

$$T_2(0) = \frac{2468}{100} \approx 24.7$$

(e) $\lim_{t \to \infty} T_2 = \dfrac{155}{2} = 77.5$

(f) The limiting temperature is 77.5.
T_1 has no horizontal asymptote.

70. $N = \dfrac{68{,}436.82 + 4731.82t}{1000 - 23.37t + 0.16t^2}, \quad 0 \le t \le 90$

(a)

(b) For 1975, $t = 75$ and $N \approx 2875$.

(c) The maximum number of graduates occured in 1978 ($t = 78$).

(d) dN/dt was the greatest in 1959 ($t \approx 59.8$).

(e) As $t \to \infty, N \to 0$ since the degree of the denominator is greater than the degree of the numerator.

71. False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$. (See Exercise 2.)

72. False. Let $y_1 = \sqrt{x + 1}$, then $y_1(0) = 1$. Thus, $y_1' = 1/(2\sqrt{x+1})$ and $y_1'(0) = 1/2$. Finally,

$$y_1'' = -\frac{1}{4(x+1)^{3/2}} \text{ and } y_1''(0) = -\frac{1}{4}.$$

Let $p = ax^2 + bx + 1$, then $p(0) = 1$. Thus, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \ge 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/(2\sqrt{x+1}), & x > 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/(4(x+1)^{3/2}), & x > 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x, but $f(x)$ *increases* without bound.

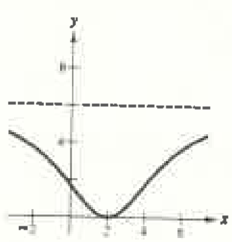
73. $x = 2$ is a critical number.

$f'(x) < 0$ for $x < 2$.

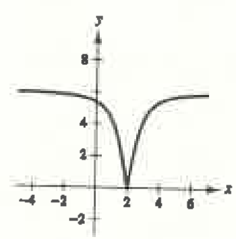
$f'(x) > 0$ for $x > 2$.

$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 6$

For example, let $f(x) = \dfrac{-6}{0.1(x-2)^2 + 1} + 6$.



74. Yes. For example, let $f(x) = \dfrac{6|x-2|}{\sqrt{(x-2)^2 + 1}}$.



75. $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m.

Case 1: If $n < m$: $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{\frac{a_n}{x^{m-n}} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{0}{b_m} = 0$

Case 2: If $m = n$: $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{a_n + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \frac{a_n}{b_m}.$

Case 3: If $n > m$: $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{a_n x^{n-m} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m}}{b_m + \cdots + \frac{b_1}{x^{m-1}} + \frac{b_0}{x^m}} = \frac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty.$

Section 3.6 A Summary of Curve Sketching

1. (a) f has constant negative slope. Matches (D)

(b) The slope of f approaches ∞ as $x \to 0^-$, and approaches $-\infty$ as $x \to 0^+$. Matches (C)

(c) The slope is periodic, and zero at $x = 0$. Matches (A)

(d) The slope is positive up to approximately $x = 1.5$. Matches (B)

2. Since the slope is negative, the function is decreasing on (2, 8), and hence $f(3) > f(5)$.

3. Since $f'(x) = 2/3$ for all x, $f(x)$ is a line of slope (2/3). The y-intercept is (0, 1): $f(x) = (2/3)x + 1$. Hence, $f(6) = 5$.

4. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

5. (a) $f'(x) = 0$ for $x = -2$ and $x = 2$

f' is negative for $-2 < x < 2$ (decreasing function).

f' is positive for $x > 2$ and $x < -2$ (increasing function).

(b) $f''(x) = 0$ at $x = 0$ (Inflection point).

f'' is positive for $x > 0$ (Concave upwards).

f'' is negative for $x < 0$ (Concave downward).

(c) f' is increasing on $(0, \infty)$. $(f'' > 0)$

(d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x.

6. $f(x) = \frac{3x^n}{x^4 + 1}$

(a) For n even, f is symmetric about the y-axis. For n odd, f is symmetric about the origin.

(b) The x-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 3$ as the horizontal asymptote.

(d) There is a slant asymptote $y = 3x$ if $n = 5$:

$$\frac{3x^5}{x^4 + 1} = 3x - \frac{3x}{x^4 + 1}.$$

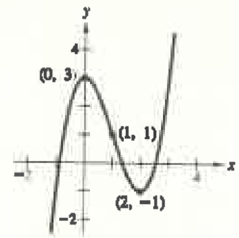
(e)

n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

7. $y = x^3 - 3x^2 + 3$

$y' = 3x^2 - 6x = 3x(x - 2) = 0$ when $x = 0, x = 2$

$y'' = 6x - 6 = 6(x - 1) = 0$ when $x = 1$

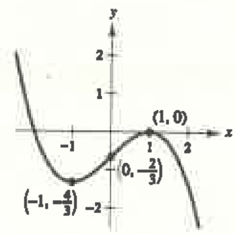


	y	y'	y''	Conclusion
$-\infty < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	3	0	$-$	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, concave down
$x = 1$	1	$-$	0	Point of inflection
$1 < x < 2$		$-$	$+$	Decreasing, concave up
$x = 2$	-1	0	$+$	Relative minimum
$2 < x < \infty$		$+$	$+$	Increasing, concave up

8. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$

$y'' = -2x = 0$ when $x = 0$



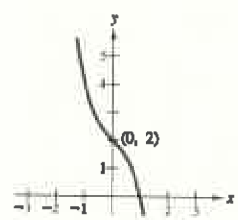
	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

9. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.



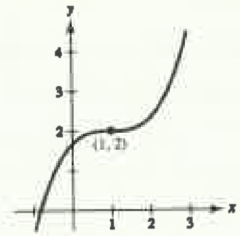
	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	2	$-$	0	Point of inflection
$0 < x < \infty$		$-$	$-$	Decreasing, concave down

10. $f(x) = \frac{1}{3}(x-1)^3 + 2$

$f'(x) = (x-1)^2 = 0$ when $x = 1$.

$f''(x) = 2(x-1) = 0$ when $x = 1$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	2	0	0	Point of inflection
$1 < x < \infty$		$+$	$+$	Increasing, concave up

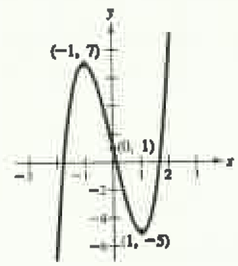


11. $f(x) = 3x^3 - 9x + 1$

$f'(x) = 9x^2 - 9 = 9(x^2 - 1) = 0$ when $x = \pm 1$

$f''(x) = 18x = 0$ when $x = 0$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		$+$	$-$	Increasing, concave down
$x = -1$	7	0	$-$	Relative maximum
$-1 < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	1	$-$	0	Point of inflection
$0 < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	-5	0	$+$	Relative minimum
$1 < x < \infty$		$+$	$+$	Increasing, concave up



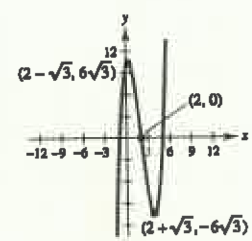
12. $f(x) = (x+1)(x-2)(x-5)$

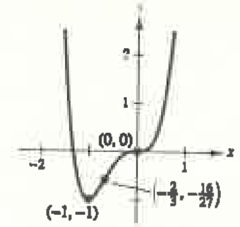
$f'(x) = (x+1)(x-2) + (x+1)(x-5) + (x-2)(x-5)$

$= 3(x^2 - 4x + 1) = 0$ when $x = 2 \pm \sqrt{3}$.

$f''(x) = 6(x-2) = 0$ when $x = 2$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 2 - \sqrt{3}$		$+$	$-$	Increasing, concave down
$x = 2 - \sqrt{3}$	$6\sqrt{3}$	0	$-$	Relative maximum
$2 - \sqrt{3} < x < 2$		$-$	$-$	Decreasing, concave down
$x = 2$	0	$-$	0	Point of inflection
$2 < x < 2 + \sqrt{3}$		$-$	$+$	Decreasing, concave up
$x = 2 + \sqrt{3}$	$-6\sqrt{3}$	0	$+$	Relative minimum
$2 + \sqrt{3} < x < \infty$		$+$	$+$	Increasing, concave up



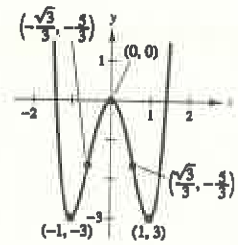


13. $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	-1	0	$+$	Relative minimum
$-1 < x < -\frac{2}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	$+$	0	Point of inflection
$-\frac{2}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		$+$	$+$	Increasing, concave up



14. $y = 3x^4 - 6x^2$

$y' = 12x^3 - 12x = 12x(x^2 - 1) = 0$ when $x = 0, x = \pm 1$.

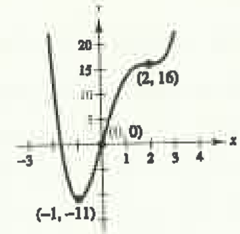
$y'' = 36x^2 - 12 = 12(3x^2 - 1) = 0$ when $x = \pm\frac{\sqrt{3}}{3}$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	-3	0	$+$	Relative minimum
$-1 < x < -\frac{\sqrt{3}}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{\sqrt{3}}{3}$	$-\frac{5}{3}$	$+$	0	Point of inflection
$-\frac{\sqrt{3}}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	0	0	$-$	Relative maximum
$0 < x < \frac{\sqrt{3}}{3}$		$-$	$-$	Decreasing, concave down
$x = \frac{\sqrt{3}}{3}$	$-\frac{5}{3}$	$-$	0	Point of inflection
$\frac{\sqrt{3}}{3} < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	-3	0	$+$	Relative minimum
$1 < x < \infty$		$+$	$+$	Increasing, concave up

15. $f(x) = x^4 - 4x^3 + 16x$

$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2 = 0$ when $x = -1, x = 2.$

$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0$ when $x = 0, x = 2.$

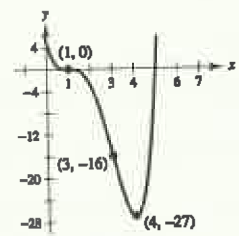


	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	-11	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 2$		$+$	$-$	Increasing, concave down
$x = 2$	16	0	0	Point of inflection
$2 < x < \infty$		$+$	$+$	Increasing, concave up

16. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

$f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2 = 0$ when $x = 1, x = 4.$

$f''(x) = 12x^2 - 48x + 36 = 12(x - 3)(x - 1) = 0$ when $x = 3, x = 1.$

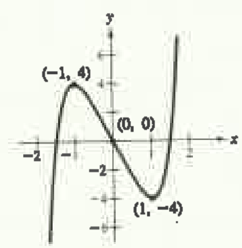


	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	0	0	0	Point of inflection
$1 < x < 3$		$-$	$-$	Decreasing, concave down
$x = 3$	-16	$-$	0	Point of inflection
$3 < x < 4$		$-$	$+$	Decreasing, concave up
$x = 4$	-27	0	$+$	Relative minimum
$4 < x < \infty$		$+$	$+$	Increasing, concave up

17. $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 0$ when $x = \pm 1.$

$y'' = 20x^3 = 0$ when $x = 0.$

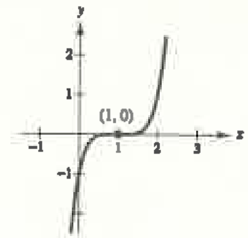


	y	y'	y''	Conclusion
$-\infty < x < -1$		$+$	$-$	Increasing, concave down
$x = -1$	4	0	$-$	Relative maximum
$-1 < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	0	$-$	0	Point of inflection
$0 < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	-4	0	$+$	Relative minimum
$1 < x < \infty$		$+$	$+$	Increasing, concave up

18. $y = (x - 1)^5$

$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.

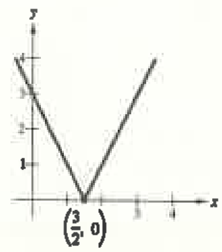


	y	y'	y''	Conclusion
$-\infty < x < 1$		+	−	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		+	+	Increasing, concave up

19. $y = |2x - 3|$

$y' = \dfrac{2(2x - 3)}{|2x - 3|}$ undefined at $x = \dfrac{3}{2}$.

$y'' = 0$



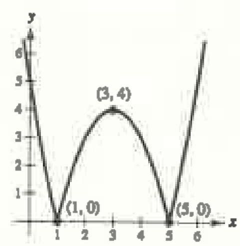
	y	y'	Conclusion
$-\infty < x < \frac{3}{2}$		−	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		+	Increasing

20. $y = |x^2 - 6x + 5|$

$y' = \dfrac{2(x - 3)(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \dfrac{2(x - 3)(x - 5)(x - 1)}{|(x - 5)(x - 1)|}$

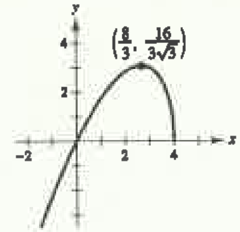
$= 0$ when $x = 3$ and undefined when $x = 1, x = 5$.

$y'' = \dfrac{2(x^2 - 6x + 5)}{|x^2 - 6x + 5|} = \dfrac{2(x - 5)(x - 1)}{|(x - 5)(x - 1)|}$ undefined when $x = 1, x = 5$.



	y	y'	y''	Conclusion
$-\infty < x < 1$		−	+	Decreasing, concave up
$x = 1$	0	Undefined	Undefined	Relative minimum, point of inflection
$1 < x < 3$		+	−	Increasing, concave down
$x = 3$	4	0	−	Relative maximum
$3 < x < 5$		−	−	Decreasing, concave down
$x = 5$	0	Undefined	Undefined	Relative minimum, point of inflection
$5 < x < \infty$		+	+	Increasing, concave up

21. $y = x\sqrt{x-4}$,



Domain: $(-\infty, 4]$

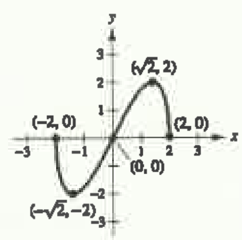
$y' = \dfrac{8-3x}{2\sqrt{4-x}} = 0$ when $x = \dfrac{8}{3}$ and undefined when $x = 4$.

$y'' = \dfrac{3x-16}{4(4-x)^{3/2}} = 0$ when $x = \dfrac{16}{3}$ and undefined when $x = 4$.

Note: $x = \frac{16}{3}$ is not in the domain.

	y	y'	y''	Conclusion
$-\infty < x < \frac{8}{3}$		$+$	$-$	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	$-$	Relative maximum
$\frac{8}{3} < x < 4$		$-$	$-$	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint

22. $y = x\sqrt{4-x^2}$,



Domain: $[-2, 2]$

$y' = \dfrac{4-2x^2}{\sqrt{4-x^2}} = 0$ when $x = \pm\sqrt{2}$ and undefined when $x = \pm 2$.

$y'' = \dfrac{2x(x^2-6)}{(4-x^2)^{3/2}} = 0$ when $x = 0, \pm 6$ and undefined when $x = \pm 2$.

Note: $x = \pm\sqrt{6}$ are not in the domain.

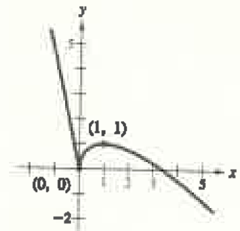
	y	y'	y''	Conclusion
$x = -2$	0	Undefined	Undefined	Endpoint
$-2 < x < -\sqrt{2}$		$-$	$+$	Decreasing, concave up
$x = -\sqrt{2}$	-2	0	$+$	Relative minimum
$-\sqrt{2} < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < \sqrt{2}$		$+$	$-$	Increasing, concave down
$x = \sqrt{2}$	2	0	$-$	Relative maximum
$\sqrt{2} < x < 2$		$-$	$-$	Decreasing, concave down
$x = 2$	0	Undefined	Undefined	Endpoint

23. $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1 - x^{1/3})}{x^{1/3}}$$

$$= 0 \text{ when } x = 1 \text{ and undefined when } x = 0.$$

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \neq 0.$$

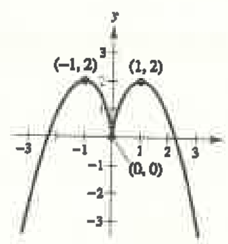


	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	1	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

24. $y = 3x^{2/3} - x^2$

$$y' = \frac{2}{x^{1/3}} - 2x = \frac{2(1 - x^{4/3})}{x^{1/3}} = 0 \text{ when } x = \pm 1, \text{ undefined when } x = 0.$$

$$y'' = -2\left(\frac{1}{3x^{4/3}} + 1\right) < 0 \text{ when } x \neq 0.$$

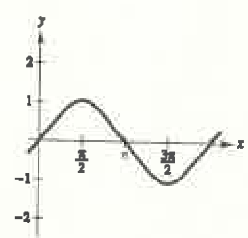


	y	y'	y''	Conclusion
$-\infty < x < -1$		$+$	$-$	Increasing, concave down
$x = -1$	2	0	$-$	Relative maximum
$-1 < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	2	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

25. $y = \sin x - \frac{1}{18}\sin 3x, \ 0 \le x \le 2\pi$

$$y' = \cos x - \frac{1}{6}\cos 3x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$y'' = -\sin x + \frac{1}{2}\sin 3x = 0 \text{ when } x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$$

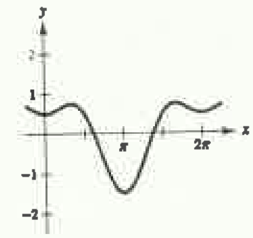


Relative maximum: $\left(\frac{\pi}{2}, \frac{19}{18}\right)$

Relative minimum: $\left(\frac{3\pi}{2}, -\frac{19}{18}\right)$

Inflection points: $\left(\frac{\pi}{6}, \frac{4}{9}\right), \left(\frac{5\pi}{6}, \frac{4}{9}\right), (\pi, 0), \left(\frac{7\pi}{6}, -\frac{4}{9}\right), \left(\frac{11\pi}{6}, -\frac{4}{9}\right)$

26. $y = \cos x - \frac{1}{2}\cos 2x, 0 \le x \le 2\pi$



$y' = -\sin x + \sin 2x = -\sin x(1 - 2\cos x) = 0$ when $x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$.

$y'' = -\cos x + 2\cos 2x = -\cos x + 2(2\cos^2 x - 1)$

$= 4\cos^2 x - \cos x - 2 = 0$ when $\cos x = \frac{1 \pm \sqrt{33}}{8} \approx 0.8431, -0.5931$.

Therefore, $x \approx 0.5678$ or 5.7154, $x \approx 2.2057$ or 4.0775.

Relative maxima: $\left(\frac{\pi}{3}, \frac{3}{4}\right), \left(\frac{5\pi}{3}, \frac{3}{4}\right)$

Relative minimum: $\left(\pi, -\frac{3}{2}\right)$

Inflection points: $(0.5678, 0.6323), (2.2057, -0.4449), (5.7154, 0.6323), (4.0775, -0.4449)$

27. $y = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

$y' = 2 - \sec^2 x = 0$ when $x = \pm\frac{\pi}{4}$.

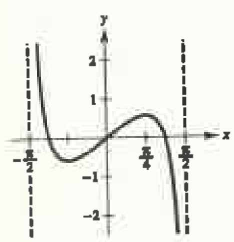
$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

Relative maximum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm\frac{\pi}{2}$



28. $y = 2x + \cot x, 0 < x < \pi$

$y' = 2 - \csc^2 x = 0$ when $x = \frac{\pi}{4}, \frac{3\pi}{4}$.

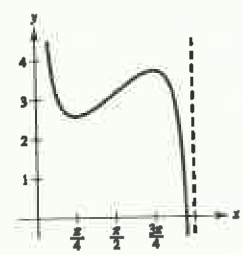
$y'' = 2\csc^2 x \cot x = 0$ when $x = \frac{\pi}{2}$.

Relative maximum: $\left(\frac{3\pi}{4}, \frac{3\pi}{2} - 1\right)$

Relative minimum: $\left(\frac{\pi}{4}, \frac{\pi}{2} + 1\right)$

Inflection point: $\left(\frac{\pi}{2}, \pi\right)$

Vertical asymptotes: $x = 0, \pi$

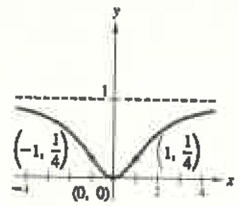


29. $y = \dfrac{x^2}{x^2 + 3}$

$y' = \dfrac{6x}{(x^2 + 3)^2} = 0$ when $x = 0$.

$y'' = \dfrac{18(1 - x^2)}{(x^2 + 3)^3} = 0$ when $x = \pm 1$.

Horizontal asymptote: $y = 1$



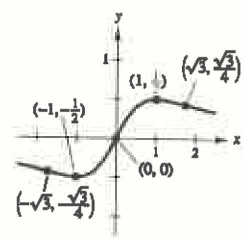
	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$-$	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	$-$	0	Point of inflection
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < 1$		$+$	$+$	Increasing, concave up
$x = 1$	$\frac{1}{4}$	$+$	0	Point of inflection
$1 < x < \infty$		$+$	$-$	Increasing, concave down

30. $y = \dfrac{x}{x^2 + 1}$

$y' = \dfrac{1 - x^2}{(x^2 + 1)^2} = \dfrac{(1 - x)(x + 1)}{(x^2 + 1)^2} = 0$ when $x = \pm 1$.

$y'' = -\dfrac{2x(3 - x^2)}{(x^2 + 1)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

Horizontal asymptote: $y = 0$



	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = -\sqrt{3}$	$-\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$-\sqrt{3} < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{1}{2}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	$\frac{1}{2}$	0	$-$	Relative maximum
$1 < x < \sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = \sqrt{3}$	$\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$\sqrt{3} < x < \infty$		$-$	$+$	Decreasing, concave up

31. $y = \dfrac{1}{x-2} - 3$

$y' = -\dfrac{1}{(x-2)^2} < 0$ when $x \neq 2$.

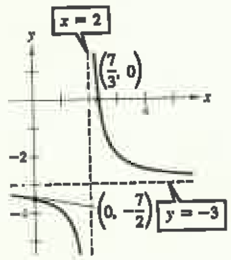
$y'' = \dfrac{2}{(x-2)^3}$

No relative extrema, no points of inflection

Intercepts: $\left(\dfrac{7}{3}, 0\right), \left(0, -\dfrac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$



32. $y = \dfrac{x^2+1}{x^2-2}$

$y' = \dfrac{-6x}{(x^2-2)^2} = 0$ when $x = 0$.

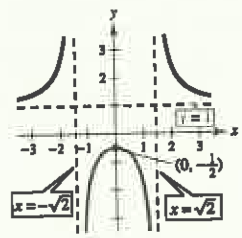
$y'' = \dfrac{6(3x^2+2)}{(x^2-2)^3} < 0$ when $x = 0$.

Therefore, $\left(0, -\dfrac{1}{2}\right)$ is a relative maximum.

Intercept: $\left(0, -\dfrac{1}{2}\right)$

Vertical asymptotes: $x = \pm\sqrt{2}$

Horizontal asymptote: $y = 1$



33. $y = \dfrac{2x}{x^2-1}$

$y' = \dfrac{-2(x^2+1)}{(x^2-1)^2} < 0$ if $x \neq \pm 1$.

$y'' = \dfrac{4x(x^2+3)}{(x^2-1)^3} = 0$ if $x = 0$.

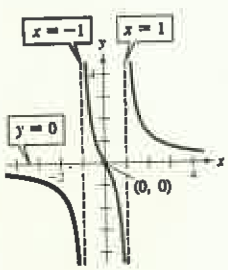
Inflection point: $(0, 0)$

Intercept: $(0, 0)$

Vertical asymptote: $x = \pm 1$

Horizontal asymptote: $y = 0$

Symmetry with respect to the origin



34. $f(x) = \dfrac{x+2}{x} = 1 + \dfrac{2}{x}$

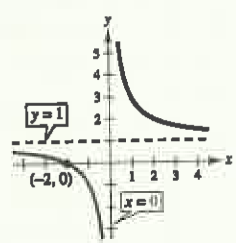
$f'(x) = \dfrac{-2}{x^2} < 0$ when $x \neq 0$.

$f''(x) = \dfrac{4}{x^3} \neq 0$

Intercept: $(-2, 0)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$



35. $g(x) = x + \dfrac{4}{x^2+1}$

$g'(x) = 1 - \dfrac{8x}{(x^2+1)^2} = \dfrac{x^4+2x^2-8x+1}{(x^2+1)^2} = 0$ when $x \approx 0.1292, 1.6085$

$g''(x) = \dfrac{8(3x^2-1)}{(x^2+1)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$

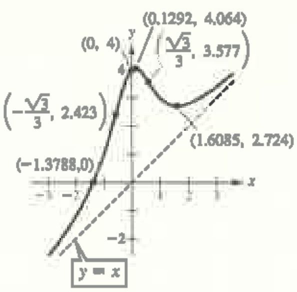
$g''(0.1292) < 0$, therefore, $(0.1292, 4.064)$ is relative maximum.

$g''(1.6085) > 0$, therefore, $(1.6085, 2.724)$ is a relative minimum.

Points of inflection: $\left(-\dfrac{\sqrt{3}}{3}, 2.423\right), \left(\dfrac{\sqrt{3}}{3}, 3.577\right)$

Intercepts: $(0, 4), (-1.3788, 0)$

Slant asymptote: $y = x$



36. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x-4)(x^2+4x+16)}{x^3} = 0$ when $x = 4$.

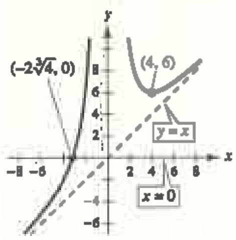
$f''(x) = \dfrac{192}{x^4} > 0$ if $x \neq 0$.

Therefore, $(4, 6)$ is a relative minimum.

Intercept; $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



37. $f(x) = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$

$f'(x) = 1 - \dfrac{1}{x^2} = 0$ when $x = \pm 1$.

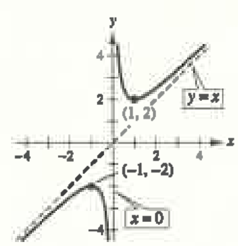
$f''(x) = \dfrac{2}{x^3} \neq 0$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



38. $f(x) = \dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1}$

$f'(x) = \dfrac{x^2(x^2-3)}{(x^2-1)^2} = 0$ when $x = 0, \pm\sqrt{3}$.

$f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3} = 0$ when $x = 0$.

Relative maximum: $\left(-\sqrt{3}, -\dfrac{3\sqrt{3}}{2}\right)$

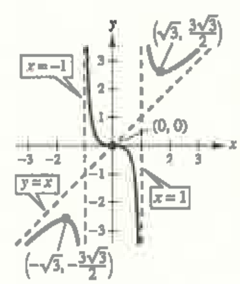
Relative minimum: $\left(\sqrt{3}, \dfrac{3\sqrt{3}}{2}\right)$

Inflection point: $(0, 0)$

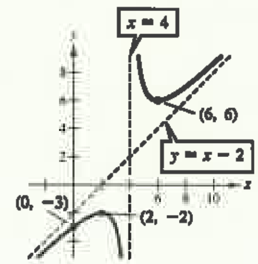
Vertical asymptotes: $x = \pm 1$

Slant asymptote: $y = x$

Origin symmetry



39. $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$



$y' = 1 - \dfrac{4}{(x - 4)^2}$

$= \dfrac{(x - 2)(x - 6)}{(x - 4)^2} = 0$ when $x = 2, 6$.

$y'' = \dfrac{8}{(x - 4)^3}$

$y'' < 0$ when $x = 2$.

Therefore, $(2, -2)$ is a relative maximum.

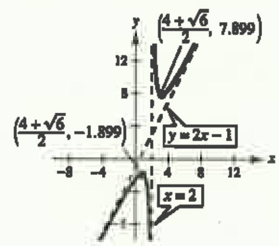
$y'' > 0$ when $x = 6$.

Therefore, $(6, 6)$ is a relative minimum.

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

40. $y = \dfrac{2x^2 - 5x + 5}{x - 2} = 2x - 1 + \dfrac{3}{x - 2}$



$y' = 2 - \dfrac{3}{(x - 2)^2} = \dfrac{2x^2 - 8x + 5}{(x - 2)^2} = 0$ when $x = \dfrac{4 \pm \sqrt{6}}{2}$

$y'' = \dfrac{6}{(x - 2)^3} \neq 0$

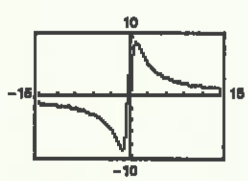
Relative maximum: $\left(\dfrac{4 - \sqrt{6}}{2}, -1.8990\right)$

Relative minimum: $\left(\dfrac{4 + \sqrt{6}}{2}, 7.8990\right)$

Vertical asymptote: $x = 2$

Slant asymptote: $y = 2x - 1$

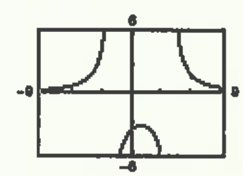
41. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2 + 1)}$



$x = 0$ vertical asymptote

$y = 0$ horizontal asymptote

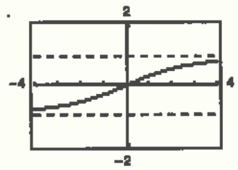
42. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$



$x = -2, 4$ vertical asymptote

$y = 0$ horizontal asymptote

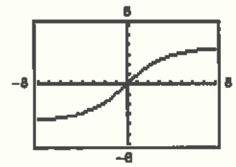
43. $y = \dfrac{x}{\sqrt{x^2 + 7}}$



(0, 0) point of inflection

$y = \pm 1$ horizontal asymptotes

44. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$



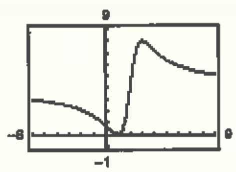
$y = \pm 4$ horizontal asymptotes

(0, 0) point of inflection

45. $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

Horizontal asymptote: $y = 4$

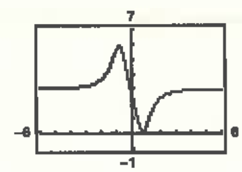


The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

46. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none

Horizontal asymptote: $y = 3$

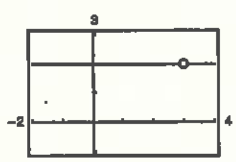


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

47. $h(x) = \dfrac{6 - 2x}{3 - x}$

$$= \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

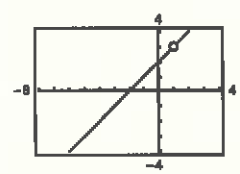


hole at (3, 2)

48. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

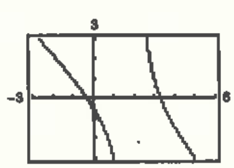
$$= \frac{(x+2)(x-1)}{x-1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



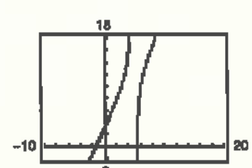
hole at (1, 3)

49. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2} = -x + 1 + \dfrac{3}{x - 2}$



The graph appears to approach the slant asymptote $y = -x + 1$.

50. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$



The graph appears to approach the slant asymptote $y = 2x + 2$.

51. Vertical asymptote: $x = 5$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x-5}$$

52. Vertical asymptote: $x = -3$

Horizontal asymptote: none

$$y = \frac{x^2}{x+3}$$

53. Vertical asymptote: $x = 5$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x-5} = \frac{3x^2 - 13x - 9}{x-5}$$

54. Vertical asymptote: $x = 0$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x} = \frac{1-x^2}{x}$$

55. $f(x) = \dfrac{ax}{(x-b)^2}$

(a) The graph has a vertical asymptote at $x = b$. If $a > 0$, the graph approaches ∞ as $x \to b$. If $a < 0$, the graph approaches $-\infty$ as $x \to b$. The graph approaches its vertical asymptote faster as $|a| \to 0$.

(b) As b varies, the position of the vertical asymptote changes: $x = b$. Also, the coordinates of the minimum are changed.

56. $f(x) = \dfrac{1}{2}(ax)^2 - (ax) = \dfrac{1}{2}(ax)(ax-2), a \neq 0$

$f'(x) = a^2x - a = a(ax-1) = 0$ when $x = \dfrac{1}{a}$.

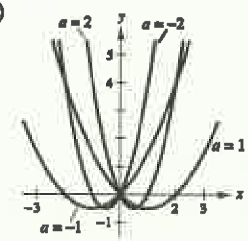
$f''(x) = a^2 > 0$ for all x.

(a) Intercepts: $(0, 0), \left(\dfrac{2}{a}, 0\right)$

Relative minimum: $\left(\dfrac{1}{a}, -\dfrac{1}{2}\right)$

Points of inflection: none

(b)



57. $f'(x) = -(x-1)(x-3) = -x^2 + 4x - 3$

Relative minimum when $x = 1$.

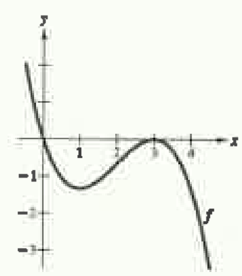
Relative maximum when $x = 3$.

$$f(x) = -\frac{x^3}{3} + 2x^2 - 3x + C, C \text{ is any constant}.$$

Let $C = 0; f(x) = -\dfrac{x^3}{3} + 2x^2 - 3x$

$$f(1) = -\frac{4}{3}$$

$$f(3) = 0.$$

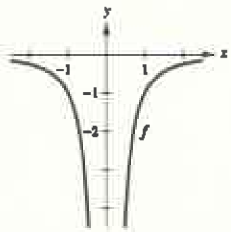


58. $f' > 0$ for $x > 0 \Rightarrow f(x)$ is increasing.

$f' < 0$ for $x < 0 \Rightarrow f(x)$ is decreasing.

f' is undefined when $x = 0$.

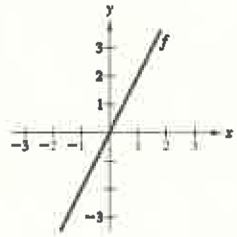
One such function is $f(x) = -\dfrac{1}{x^2}$.



59. $f'(x) = 2$

$f(x) = 2x + C$, C is any constant.

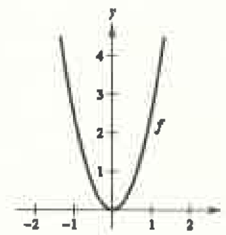
If $C = 0, f(x) = 2x$.



60. f' is linear $\Rightarrow f$ is quadratic.
$f' < 0$ when $x < 0 \Rightarrow f$ is decreasing. $f' > 0$ when $x > 0 \Rightarrow f$ is increasing.
Therefore, $(0, f(0))$ is a relative minimum.

$f(x) = ax^2 + b, a > 0$

If $a = \frac{5}{2}$ and $b = 0$, then $f(x) = \frac{5}{2}x^2$.

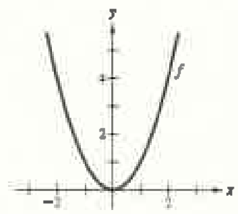


61. $f''(x) = 2 > 0 \Rightarrow f$ is concave up.

$f'(x) = 2x + C_1$

$f(x) = x^2 + C_1x + C_2$

Let $C_1 = C_2 = 0$, then $f(x) = x^2$.



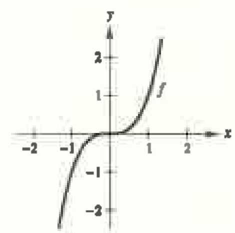
62. f'' is linear $\Rightarrow f$ is cubic.

$f(x) = ax^3 + bx^2 + cx + d$

$f'' < 0$ when $x < 0 \Rightarrow$ concave down.

$f'' > 0$ when $x > 0 \Rightarrow$ concave up.

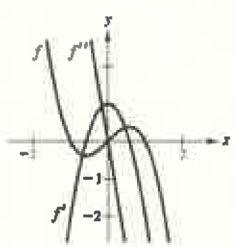
Therefore, $(0, f(0))$ is a point of inflection. Since f is concave up to the right and concave down to the left $\Rightarrow a > 0$. Let $f(x) = x^3$.



63. f is cubic.

f' is quadratic.

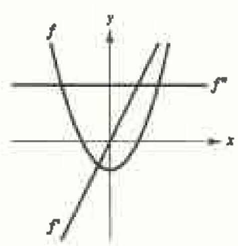
f'' is linear.



64. f'' is constant.

f' is linear.

f'' is quadratic.



In Exercises 65–70

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f''(x) = 6ax + 2b$$

$$f'(x) = 0 \text{ when } x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

Furthermore,

$\lim_{x \to \infty} f(x) = \infty$ if and only if $a > 0$.

$\lim_{x \to \infty} f(x) = -\infty$ if and only if $a < 0$.

65. Since $\lim_{x \to \infty} f(x) = -\infty$, $a < 0$.

Also, $f'(x) < 0$ for all x. Therefore, the discriminant is negative.

$b^2 - 3ac < 0 \Rightarrow b^2 < 3ac$

66. Since $\lim_{x \to \infty} f(x) = \infty$, $a > 0$.

Also, $f'(x) > 0$ for all x. Therefore, the discrimanant is negative.

$b^2 - 3ac < 0 \Rightarrow b^2 < 3ac$

67. Since $\lim_{x \to \infty} f(x) = -\infty$, $a < 0$.

Also, since there is only one critical point, the discriminant is zero.

$b^2 - 3ac = 0 \Rightarrow b^2 = 3ac$

68. Since $\lim_{x \to \infty} f(x) = \infty$, $a > 0$.

Also, since there is only one critical point, the discriminant is zero.

$b^2 - 3ac = 0 \Rightarrow b^2 = 3ac$

69. Since $\lim_{x \to \infty} f(x) = -\infty$, $a < 0$.

Also, since there are two critical points, the discriminant is positive.

$b^2 - 3ac > 0 \Rightarrow b^2 > 3ac$

70. Since $\lim_{x \to \infty} f(x) = \infty$, $a > 0$.

Also, since there are two critical points, the discriminant is positive.

$b^2 - 3ac > 0 \Rightarrow b^2 > 3ac$

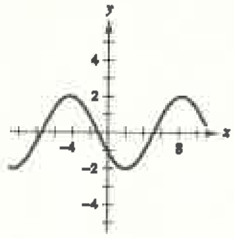
71.

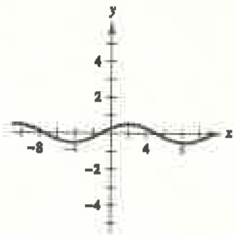
(any vertical translate of f will do)

72.

(any vertical translate of f will do)

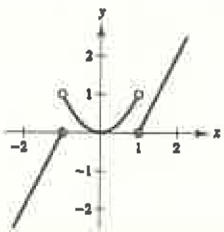
73.

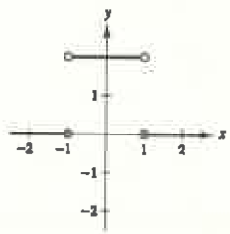




(any vertical translate of f will do)

74.





(any vertical translate of the 3 segments of f will do)

Section 3.7 Optimization Problems

1. (a)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

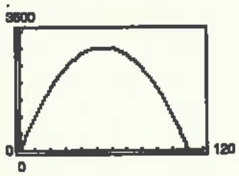
The maximum is attained near $x = 50$ and 60.

—CONTINUED—

1. —CONTINUED—

(c) $P = x(110 - x) = 110x - x^2$

(d)



The solution appears to be $x = 55$.

(e) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$\dfrac{d^2P}{dx^2} = -2 < 0$

P is a maximum when $x = 110 - x = 55$.

2. Let x and y be two positive numbers such that $x + y = S$.

$$P = xy = x(S - x) = Sx - x^2$$

$$\frac{dP}{dx} = S - 2x = 0 \text{ when } x = \frac{S}{2}.$$

$$\frac{d^2P}{dx^2} = -2 < 0 \text{ when } x = \frac{S}{2}.$$

P is a maximum when $x = y = S/2$.

3. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + y = x + \frac{192}{x}$$

$$\frac{dS}{dx} = 1 - \frac{192}{x^2} = 0 \text{ when } x = \sqrt{192}.$$

$$\frac{d^2S}{dx^2} = \frac{384}{x^3} > 0 \text{ when } x = \sqrt{192}.$$

S is a minimum when $x = y = \sqrt{192}$.

4. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + 3y = \frac{192}{y} + 3y$$

$$\frac{dS}{dy} = 3 - \frac{192}{y^2} = 0 \text{ when } y = 8.$$

$$\frac{d^2S}{dy^2} = \frac{384}{y^3} > 0 \text{ when } y = 8.$$

S is minimum when $y = 8$ and $x = 24$.

5. Let x be a positive number.

$$S = x + \frac{1}{x}$$

$$\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0 \text{ when } x = 1.$$

$$\frac{d^2S}{dx^2} = \frac{2}{x^3} > 0 \text{ when } x = 1.$$

The sum is a minimum when $x = 1$ and $1/x = 1$.

6. Let x and y be two positive numbers such that $x + 2y = 100$.

$$P = xy = y(100 - 2y) = 100y - 2y^2$$

$$\frac{dP}{dy} = 100 - 4y = 0 \text{ when } y = 25.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 25.$$

P is a maximum when $x = 50$ and $y = 25$.

7. Let x be the length and y the width of the rectangle.

$$2x + 2y = 100$$

$$y = 50 - x$$

$$A = xy = x(50 - x)$$

$$\frac{dA}{dx} = 50 - 2x = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 25.$$

A is maximum when $x = y = 25$ meters.

8. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$

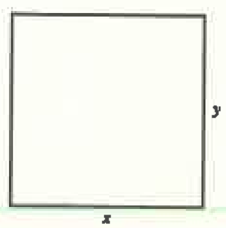
$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)



9. Let x be the length and y the width of the rectangle.

$$xy = 64$$

$$y = \frac{64}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$$\frac{dP}{dx} = 2 - \frac{128}{x^2} = 0 \text{ when } x = 8.$$

$$\frac{d^2P}{dx^2} = \frac{256}{x^3} > 0 \text{ when } x = 8.$$

P is minimum when $x = y = 8$ feet.

10. Let x be the length and y the width of the rectangle.

$$xy = A$$

$$y = \frac{A}{x}$$

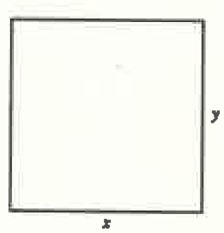
$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

P is minimum when $x = y = \sqrt{A}$ centimeters.

(A square!)



11. $d = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$

$= \sqrt{x^2 - 7x + 16}$

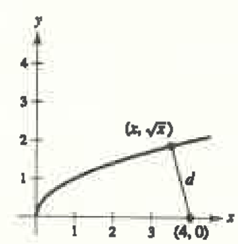
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$

$$f'(x) = 2x - 7 = 0$$

$$x = \frac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $(7/2, \sqrt{7/2})$.



12. $d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$

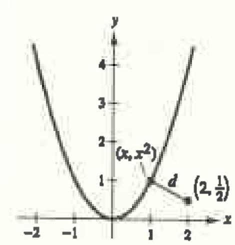
$= \sqrt{x^4 - 4x + (17/4)}$.

Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 4x + \frac{17}{4}$.

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.



13. $\dfrac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$

$\dfrac{d^2Q}{dx^2} = kQ_0 - 2kx$

$= k(Q_0 - 2x) = 0$ when $x = \dfrac{Q_0}{2}$.

$\dfrac{d^3Q}{dx^3} = -2k < 0$ when $x = \dfrac{Q_0}{2}$.

dQ/dx is maximum when $x = Q_0/2$.

14. $F = \dfrac{v}{22 + 0.02v^2}$

$\dfrac{dF}{dv} = \dfrac{22 - 0.02v^2}{(22 + 0.02v^2)^2}$

$= 0$ when $v = \sqrt{1100} \approx 33.166$.

By the First Derivative Test, the flow rate on the road is maximized when $v \approx 33$ mph.

15. $xy = 180{,}000$ (see figure)

$S = x + 2y = \left(x + \dfrac{360{,}000}{x}\right)$ where S is the length of fence needed.

$\dfrac{dS}{dx} = 1 - \dfrac{360{,}000}{x^2} = 0$ when $x = 600$.

$\dfrac{d^2S}{dx^2} = \dfrac{720{,}000}{x^3} > 0$ when $x = 600$.

S is a minimum when $x = 600$ meters and $y = 300$ meters.

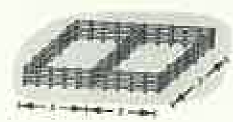
16. $4x + 3y = 200$ is the perimeter. (see figure)

$A = 2xy = 2x\left(\dfrac{200 - 4x}{3}\right) = \dfrac{8}{3}(50x - x^2)$

$\dfrac{dA}{dx} = \dfrac{8}{3}(50 - 2x) = 0$ when $x = 25$.

$\dfrac{d^2A}{dx^2} = -\dfrac{16}{3} < 0$ when $x = 25$.

A is a maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.



17. (a) $A = 4(\text{area of side}) + 2(\text{area of Top})$

(a) $A = 4(3)(11) + 2(3)(3) = 150$ square inches

(b) $A = 4(5)(5) + 2(5)(5) = 150$ square inches

(c) $A = 4(3.25)(6) + 2(6)(6) = 150$ square inches

(b) $V = (\text{length})(\text{width})(\text{height})$

(a) $V = (3)(3)(11) = 99$ cubic inches

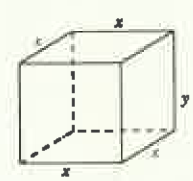
(b) $V = (5)(5)(5) = 125$ cubic inches

(c) $V = (6)(6)(3.25) = 117$ cubic inches

(c) $S = 4xy + 2x^2 = 150 \Longrightarrow y = \dfrac{150 - 2x^2}{4x}$

$V = x^2y = x^2\left(\dfrac{150 - 2x^2}{4x}\right) = \dfrac{75}{2}x - \dfrac{1}{2}x^3$

$V' = \dfrac{75}{2} - \dfrac{3}{2}x^2 = 0 \Longrightarrow x = \pm 5$



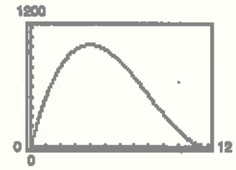
By the First Derivative Test, $x = 5$ yields the maximum volume. Dimensions: $5 \times 5 \times 5$. (A cube!)

18. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

(b) $V = x(24 - 2x)^2, 0 < x < 12$

(d)



The maximum volume seems to be 1024.

(c) $\dfrac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$

$= 12(12 - x)(4 - x) = 0$ when $x = 12, 4$ (12 is not in the domain).

$\dfrac{d^2V}{dx^2} = 12(2x - 16)$

$\dfrac{d^2V}{dx^2} < 0$ when $x = 4$.

When $x = 4$, $V = 1024$ is maximum.

19. (a) $V = x(s - 2x)^2, 0 < x < \dfrac{s}{2}$

$\dfrac{dV}{dx} = 2x(s - 2x)(-2) + (s - 2x)^2$

$= (s - 2x)(s - 6x) = 0$ when $x = \dfrac{s}{2}, \dfrac{s}{6}$ ($s/2$ is not in the domain).

$\dfrac{d^2V}{dx^2} = 24x - 8s$

$\dfrac{d^2V}{dx^2} < 0$ when $x = \dfrac{s}{6}$.

$V = \dfrac{2s^3}{27}$ is maximum when $x = \dfrac{5}{6}$.

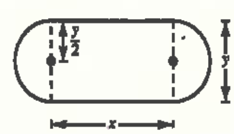
(b) If the length is doubled, $V = \frac{2}{27}(2s)^3 = 8\left(\frac{2}{27}s^3\right)$. Volume is increased by a factor of 8.

20. (a) $P = 2x + 2\pi r$

$= 2x + 2\pi\left(\dfrac{y}{2}\right)$

$= 2x + \pi y = 200$

$\Rightarrow y = \dfrac{200 - 2x}{\pi} = \dfrac{2}{\pi}(100 - x)$



—CONTINUED—

20. —CONTINUED—

(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100-10)$	$(10)\frac{2}{\pi}(100-10) \approx 573$
20	$\frac{2}{\pi}(100-20)$	$(20)\frac{2}{\pi}(100-20) \approx 1019$
30	$\frac{2}{\pi}(100-30)$	$(30)\frac{2}{\pi}(100-30) \approx 1337$
40	$\frac{2}{\pi}(100-40)$	$(40)\frac{2}{\pi}(100-40) \approx 1528$
50	$\frac{2}{\pi}(100-50)$	$(50)\frac{2}{\pi}(100-50) \approx 1592$
60	$\frac{2}{\pi}(100-60)$	$(60)\frac{2}{\pi}(100-60) = 1528$

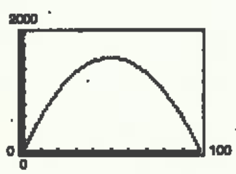
The maximum area of the rectangle is approximately 1592 m^2.

(c) $A = xy = x\frac{2}{\pi}(100-x) = \frac{2}{\pi}(100x - x^2)$

(d) $\frac{dA}{dx} = \frac{2}{\pi}(100-2x) = 0 \Longrightarrow x = 50.$

By the First Derivative Test, this yields the maximum area, $A = 5000/\pi \approx 1591.55$ m^2.

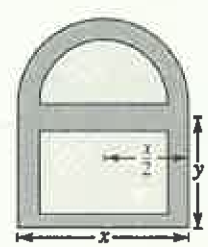
(e)



Maximum area is approximately

1591.55 m^2 ($x = 50$ m).

21. $16 = 2y + x + \pi\left(\frac{x}{2}\right)$



$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

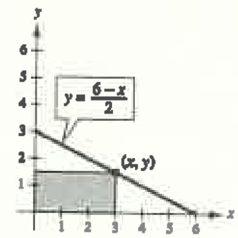
$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4+\pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4+\pi}$$

$$y = \frac{32 - 2[32/(4+\pi)] - \pi[32/(4+\pi)]}{4} = \frac{16}{4+\pi}$$

The area is maximum when $y = \frac{16}{4+\pi}$ feet and $x = \frac{32}{4+\pi}$ feet.

22. You can see from the figure that $A = xy$ and $y = \dfrac{6-x}{2}$.



$$A = x\left(\frac{6-x}{2}\right) = \frac{1}{2}(6x - x^2).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

A is a maximum when $x = 3$ and $y = 3/2$.

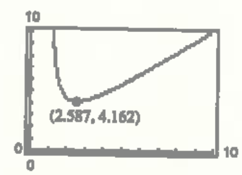
23. (a) $\dfrac{y-2}{0-1} = \dfrac{0-2}{x-1}$

$$y = 2 + \frac{2}{x-1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x-1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x-1} + \frac{4}{(x-1)^2}}, \quad x > 1$$

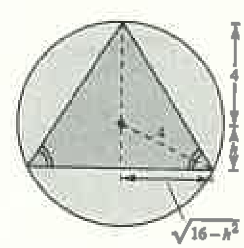
(b)



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

(c) Vertices: $(0, 0)$, $(2.587, 0)$ and $(0, 3.260)$

24. $A = \dfrac{1}{2}\text{base} \times \text{height}$



$$= \frac{1}{2}\left(2\sqrt{16-h^2}\right)(4 + h) = \sqrt{16-h^2}(4 + h)$$

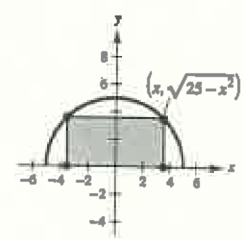
$$\frac{dA}{dh} = \frac{1}{2}(16 - h^2)^{-1/2}(-2h)(4 + h) + (16 - h^2)^{1/2}$$

$$= (16 - h^2)^{-1/2}[-h(4 + h) + (16 - h^2)]$$

$$= \frac{-2[h^2 + 2h - 8]}{\sqrt{16-h^2}} = \frac{-2(h+4)(h-2)}{\sqrt{16-h^2}}$$

$dA/dh = 0$ when $h = 2$, which is a maximum by the First Derivative Test. Hence, the sides of the equilateral triangle are $2\sqrt{16-h^2} = 2\sqrt{16-4} = 4\sqrt{3}$.

25. $A = 2xy = 2x\sqrt{25-x^2}$ (see figure)



$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25-x^2}}\right) + 2\sqrt{25-x^2}$$

$$= 2\left(\frac{25-2x^2}{\sqrt{25-x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

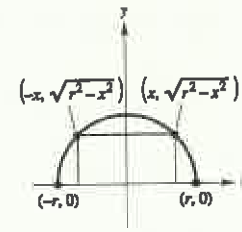
$$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

Width: $\dfrac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$

26. $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.

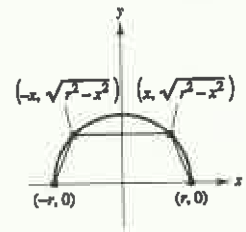


27. $A = \frac{1}{2}(2r + 2x)\sqrt{r^2 - x^2} = (r + x)\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = (r + x)\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + \sqrt{r^2 - x^2}$$

$$= \frac{-x(r + x)}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}}$$

$$= \frac{(r - 2x)(r + x)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{r}{2}.$$



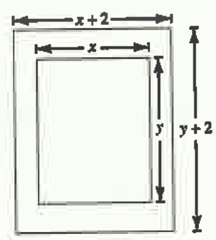
By the First Derivative Test, A will be a maximum when the trapezoid bases are r and $2r$, and the altitude is $(\sqrt{3}r)/2$.

28. $xy = 30 \Longrightarrow y = \dfrac{30}{x}$

$$A = (x + 2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$

$$\frac{dA}{dx} = (x + 2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2 - 30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$

$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$



By the First Derivative Test, the dimensions $(x + 2)$ by $(y + 2)$ are $(2 + \sqrt{30})$ by $(2 + \sqrt{30})$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.

29. $V = \pi r^2 h = 22$ cubic inches or $h = \dfrac{22}{\pi r^2}$

(a)

Radius, r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\dfrac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\dfrac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$

(b)

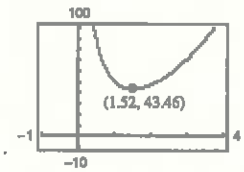
Radius, r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\dfrac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\dfrac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\dfrac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \dfrac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\dfrac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \dfrac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\dfrac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \dfrac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\dfrac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \dfrac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\dfrac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \dfrac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\dfrac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \dfrac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to be about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\dfrac{dS}{dr} = 4\pi r - \dfrac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$h = \dfrac{22}{\pi r^2} \approx 3.04$ in.

Note: Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

30. $V = \pi r^2 h = V_0$ cubic units or $h = \dfrac{V_0}{\pi r^2}$

$$S = 2\pi r^2 + 2\pi rh = 2\left(\pi r^2 + \frac{V_0}{r}\right)$$

$$\frac{dS}{dr} = 2\left(2\pi r - \frac{V_0}{r^2}\right) = 0 \text{ when } r = \sqrt[3]{\frac{V_0}{2\pi}} \text{ units.}$$

$$h = \frac{V_0}{\pi\left(\sqrt[3]{V_0/2\pi}\right)^2} = \frac{V_0(2\pi)^{2/3}}{\pi V_0^{2/3}} = \frac{2V_0^{1/3}}{(2\pi)^{1/3}} = 2r$$

By the First Derivative Test, this will yield the minimum surface area.

31. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \implies y = 108 - 4x$$

$$V = x^2 y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

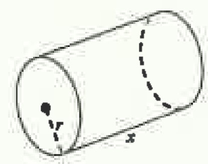
$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

32. $V = \pi r^2 x$

$$x + 2\pi r = 108 \implies x = 108 - 2\pi r \quad \text{(see figure)}$$



$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.

33. $V = \dfrac{1}{3}\pi x^2 h = \dfrac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right)\right] = \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0$$

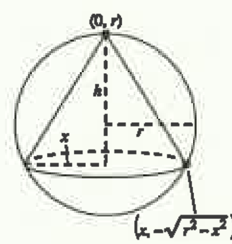
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$



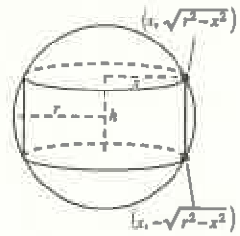
By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

34. $V = \pi x^2 h = \pi x^2\left(2\sqrt{r^2 - x^2}\right) = 2\pi x^2\sqrt{r^2 - x^2}$ (see figure)



$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}(2r^2 - 3x^2)$$

$$= 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \Longrightarrow x = \frac{\sqrt{6}r}{3}.$$

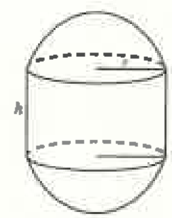
By the First Derivative Test, the volume is a maximum when

$$x = \frac{\sqrt{6}r}{3} \text{ and } h = \frac{2r}{\sqrt{3}}.$$

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$

35. $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$



$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

$$S = 4\pi r^2 + 2\pi r h = 4\pi r^2 + 2\pi r\left(\frac{12}{\pi r^2} - \frac{4}{3}r\right)$$

$$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0 \text{ when } r = \sqrt[3]{9/\pi} \approx 1.42 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0 \text{ when } r = \sqrt[3]{9/\pi} \text{ cm.}$$

The surface area is minimum when $r = \sqrt[3]{9/\pi}$ cm and $h = 0$. The resulting solid is a sphere of radius $r \approx 1.42$ cm.

36. $V = 3000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi r h) = k\left[8\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{6000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{6000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}} \approx 5.636 \text{ feet and } h \approx 22.545 \text{ feet.}$$

By the Second Derivative Test, we have

$$\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}}.$$

Therefore, these dimensions will produce a minimum cost.

37. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

A is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

38. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2,\ 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0, A = 25$, when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$, and when $x = 20/3, A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}(\sqrt{3})\left(\frac{20 - 5x}{6}\right)^2, 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0, A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4, A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon.

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2,\ 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0, A \approx 27.528$, when $x \approx 2.62, A \approx 13.102$, and when $x = 5, A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, 0 \le x \le \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0, A \approx 31.831$, when $x \approx 1.748, A \approx 15.138$, and when $x = 10/3, A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

39. Let S be the strength and k the constant of proportionality. Given $h^2 + w^2 = 24^2$, $h^2 = 24^2 - w^2$,

$$S = kwh^2$$

$$S = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^3) = 0 \text{ when } w = 8\sqrt{3}, h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

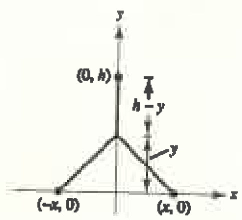
40. Let A be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.



41. $$R = \frac{v_0^2}{g}\sin 2\theta$$

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g}\cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g}\sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test, R is maximum when $\theta = \pi/4$.

42. (a) $$s = \frac{v\,\frac{\text{km}}{\text{hr}}\left(1000\,\frac{\text{m}}{\text{km}}\right)}{\left(3600\,\frac{\text{sec}}{\text{hr}}\right)} = \frac{5}{18}v$$

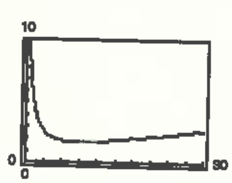
v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$d(t) = 0.071s^2 + 0.398s + 0.727$

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(t)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(t)/s$ more seconds. Hence,

$$T = \frac{d(t)}{s} + \frac{5.5}{s}.$$

(c)



$$T = \frac{1}{s}(0.071s^2 + 0.389s + 0.727) + \frac{5.5}{s}$$

The minimum is attained when $s \approx 9.365$ seconds.

(d) $$T(s) = 0.071s + 0.389 + \frac{6.227}{s}$$

$$T'(s) = 0.071 - \frac{6.227}{s^2} \Rightarrow s^2 = \frac{6.227}{0.071}$$

$$\Rightarrow s \approx 9.365 \text{ m/sec}$$

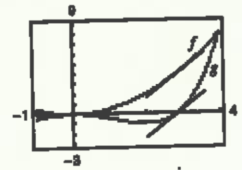
$T(9.365) \approx 1.719$ seconds

$$9.365 \text{ m/sec} \cdot \frac{3600}{1000} = 3.37 \text{ km/hr}$$

(e) $d(9.365) = 10.597$

43. $f(x) = \frac{1}{2}x^2 \qquad g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on $[0, 4]$

(a)



(b) $d(x) = f(x) - g(x) = \frac{1}{2}x^2 - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) = x^2 - \frac{1}{16}x^4$

$d'(x) = 2x - \frac{1}{4}x^3 = 0 \Longrightarrow 8x = x^3$

$\Longrightarrow x = 0, 2\sqrt{2}$ (in $[0, 4]$)

The maximum distance is $d = 4$ when $x = 2\sqrt{2}$.

(c) $f'(x) = x$, Tangent line at $(2\sqrt{2}, 4)$ is

$$y - 4 = 2\sqrt{2}(x - 2\sqrt{2})$$
$$y = 2\sqrt{2}x - 4.$$

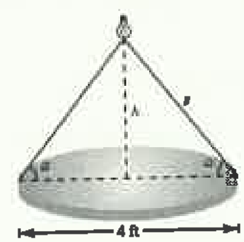
$g'(x) = \frac{1}{4}x^3 - x$, Tangent line at $(2\sqrt{2}, 0)$ is

$$y - 0 = \left(\tfrac{1}{4}(2\sqrt{2})^3 - 2\sqrt{2}\right)(x - 2\sqrt{2})$$
$$y = 2\sqrt{2}x - 8.$$

The tangent lines are parallel and 4 vertical units apart.

(d) The tangent lines will be parallel. If $d(x) = f(x) - g(x)$, then $d'(x) = 0 = f'(x) - g'(x)$ implies that $f'(x) = g'(x)$ at the point x where the distance is maximum.

44. $\sin\alpha = \dfrac{h}{s} \Longrightarrow s = \dfrac{h}{\sin\alpha},\ 0 < \alpha < \dfrac{\pi}{2}$



$$\tan\alpha = \frac{h}{2} \Longrightarrow h = 2\tan\alpha \Longrightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$$

$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4}[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)]$$

$$= \frac{k}{4}\cos\alpha[\cos^2\alpha - 2\sin^2\alpha]$$

$$= \frac{k}{4}\cos\alpha[1 - 3\sin^2\alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin\alpha = \pm\frac{1}{\sqrt{3}}.$$

Since α is acute, we have

$$\sin\alpha = \frac{1}{\sqrt{3}} \Longrightarrow h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \text{ feet.}$$

Since $(d^2I)/(d\alpha^2) = (k/4)\sin\alpha(9\sin^2\alpha - 7) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.

45. Let F be the illumination at point P which is x units from source 1.

$$F = \frac{kI_1}{x^2} + \frac{kI_2}{(d-x)^2}$$

$$\frac{dF}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(d-x)^3} = 0 \text{ when } \frac{2kI_1}{x^3} = \frac{2kI_2}{(d-x)^3}.$$

$$\frac{\sqrt[3]{I_1}}{\sqrt[3]{I_2}} = \frac{x}{d-x}$$

$$(d-x)\sqrt[3]{I_1} = x\sqrt[3]{I_2}$$

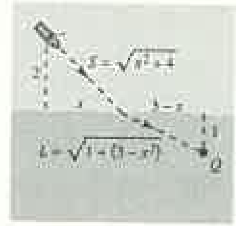
$$d\sqrt[3]{I_1} = x\left(\sqrt[3]{I_1} + \sqrt[3]{I_2}\right)$$

$$x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2F}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(d-x)^4} > 0 \text{ when } x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}.$$

This is the minimum point.

46.
$$S = \sqrt{x^2+4},\ L = \sqrt{1+(3-x)^2}$$



$$\text{Time} = T = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}} = 0$$

$$\frac{x^2}{x^2+4} = \frac{9-6x+x^2}{4(x^2-6x+10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphics calculator, you can determine that this equation has only one root in this interval ($x = 1$). Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

47. $T = \dfrac{\sqrt{x^2+4}}{v_1} + \dfrac{\sqrt{x^2-6x+10}}{v_2}$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+4}} + \frac{x-3}{v_2\sqrt{x^2-6x+10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2+4}} = \sin\theta_1 \text{ and } \frac{x-3}{\sqrt{x^2-6x+10}} = -\sin\theta_2$$

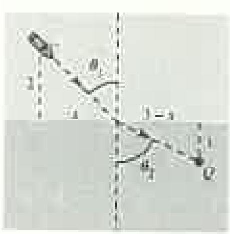
we have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Longrightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2+4)^{3/2}} + \frac{1}{v_2(x^2-6x+10)^{3/2}} > 0$$

this condition yields a minimum time.



48. $T = \dfrac{\sqrt{x^2+d_1^2}}{v_1} + \dfrac{\sqrt{d_2^2+(a-x)^2}}{v_2}$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+d_1^2}} + \frac{x-a}{v_2\sqrt{d_2^2+(a-x)^2}} = 0$$

Since

$$\frac{x}{\sqrt{x^2+d_1^2}} = \sin\theta_1 \text{ and } \frac{x-a}{\sqrt{d_2^2+(a-x)^2}} = -\sin\theta_2$$

we have

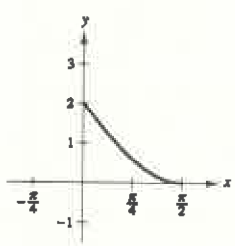
$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Longrightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2+d_1^2)^{3/2}} + \frac{d_2^2}{v_2[d_2^2+(a-x)^2]^{3/2}} > 0$$

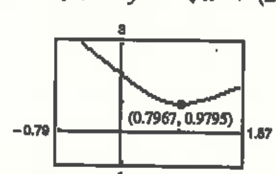
this condition yields a minimum time.

49. $f(x) = 2 - 2\sin x$



(a) Distance from origin to y-intercept is 2.
Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2+y^2} = \sqrt{x^2+(2-2\sin x)^2}$

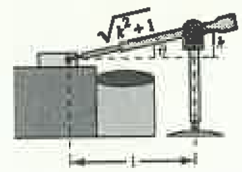


Minimum distance $= 0.9795$ at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.



50. $F\cos\theta = k(W - F\sin\theta)$

$$F = \frac{kW}{\cos\theta + k\sin\theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k\cos\theta - \sin\theta)}{(\cos\theta + k\sin\theta)^2} = 0$$

$k\cos\theta = \sin\theta \Longrightarrow k = \tan\theta \Longrightarrow \theta = \arctan k$

Since

$$\cos\theta + k\sin\theta = \frac{1}{\sqrt{k^2+1}} + \frac{k^2}{\sqrt{k^2+1}} = \sqrt{k^2+1},$$

the minimum force is

$$F = \frac{kW}{\cos\theta + k\sin\theta} = \frac{kW}{\sqrt{k^2+1}}.$$

51. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144 - r^2}$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)(144 - r^2)^{-1/2}(-2r) + 2r\sqrt{144 - r^2}\right]$$

$$= \frac{1}{3}\pi\left[\frac{288r - 3r^3}{\sqrt{144 - r^2}}\right] = \pi\left[\frac{r(96 - r^2)}{\sqrt{144 - r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone: $S = \pi(4\sqrt{6})\sqrt{(4\sqrt{6})^2 + (4\sqrt{3})^2} = 48\sqrt{6}\pi$

Area of sector: $144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}(3 - \sqrt{6}) \approx 1.153 \text{ radians or } 66^\circ$$

52. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10^\circ$	$8\sin 10^\circ$	≈ 22.1
8	$8 + 16\cos 20^\circ$	$8\sin 20^\circ$	≈ 42.5
8	$8 + 16\cos 30^\circ$	$8\sin 30^\circ$	≈ 59.7
8	$8 + 16\cos 40^\circ$	$8\sin 40^\circ$	≈ 72.7
8	$8 + 16\cos 50^\circ$	$8\sin 50^\circ$	≈ 80.5
8	$8 + 16\cos 60^\circ$	$8\sin 60^\circ$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10^\circ$	$8\sin 10^\circ$	≈ 22.1
8	$8 + 16\cos 20^\circ$	$8\sin 20^\circ$	≈ 42.5
8	$8 + 16\cos 30^\circ$	$8\sin 30^\circ$	≈ 59.7
8	$8 + 16\cos 40^\circ$	$8\sin 40^\circ$	≈ 72.7
8	$8 + 16\cos 50^\circ$	$8\sin 50^\circ$	≈ 80.5
8	$8 + 16\cos 60^\circ$	$8\sin 60^\circ$	≈ 83.1
8	$8 + 16\cos 70^\circ$	$8\sin 70^\circ$	≈ 80.7
8	$8 + 16\cos 80^\circ$	$8\sin 80^\circ$	≈ 74.0
8	$8 + 16\cos 90^\circ$	$8\sin 90^\circ$	≈ 64.0

The maximum cross-sectional area is approximately 83.1 square feet.

—CONTINUED—

52. —CONTINUED—

(c) $A = (a + b)\frac{h}{2}$

$= [8 + (8 + 16\cos\theta)]\frac{8\sin\theta}{2}$

$= 64(1 + \cos\theta)\sin\theta, 0° < \theta < 90°$

(d) $\frac{dA}{d\theta} = 64(1 + \cos\theta)\cos\theta + (-64\sin\theta)\sin\theta$

$= 64(\cos\theta + \cos^2\theta - \sin^2\theta)$

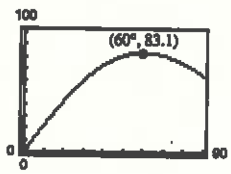
$= 64(2\cos^2\theta + \cos\theta - 1)$

$= 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$.

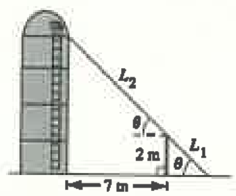
The maximum occurs when $\theta = 60°$.

(e)



53. $\sin\theta = \frac{2}{L_1} \Rightarrow L_1 = \frac{2}{\sin\theta}$

$\cos\theta = \frac{7}{L_2} \Rightarrow L_2 = \frac{7}{\cos\theta}$



(a)

θ	L_1	L_2	$L_1 + L_2$
0.1	20.0	7.0	27.1
0.2	10.1	7.1	17.2
0.3	6.8	7.3	14.1
0.4	5.1	7.6	12.7
0.5	4.2	8.0	12.1
0.6	3.5	8.5	12.0

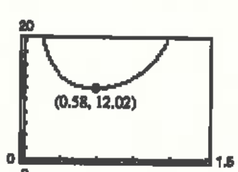
(b)

θ	L_1	L_2	$L_1 + L_2$
0.57	3.706	8.315	12.021
0.58	3.650	8.369	12.018
0.59	3.595	8.424	12.019

The minimum length of $L_1 + L_2$ is approximately 12.018 meters.

(c) $L = L_1 + L_2 = \frac{2}{\sin\theta} + \frac{7}{\cos\theta}$

(d)



The minimum is approximately 12.018 meters.

(e) $L(\theta) = \frac{2}{\sin\theta} + \frac{7}{\cos\theta} = 2\csc\theta + 7\sec\theta$

$L'(\theta) = -2\csc\theta\cot\theta + 7\sec\theta \cdot \tan\theta = 0$

$2\csc\theta\cot\theta = 7\sec\theta\tan\theta$

$2\frac{\cos\theta}{\sin^2\theta} = 7\frac{\sin\theta}{\cos^2\theta}$

$2\cos^3\theta = 7\sin^3\theta$

$\frac{2}{7} = \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta$

$\tan\theta = \left(\frac{2}{7}\right)^{1/3} = \frac{\sqrt[3]{98}}{7} \Rightarrow \theta \approx 0.5824$ radians.

$L(0.5824) \approx 12.018$ meters

(f) height $= L \cdot \sin\theta = (12.018)(\sin(0.5824))$

≈ 6.61 meters

54. $E = \dfrac{\tan \phi(1 - 0.1 \tan \phi)}{0.1 + \tan \phi}$

$$\frac{dE}{d\phi} = \frac{(0.1 + \tan \phi)[\sec^2 \phi - 0.2 \tan \phi \sec^2 \phi] - (\tan \phi - 0.1 \tan^2 \phi)(\sec^2 \phi)}{(0.1 + \tan \phi)^2}$$

$$= \frac{0.1 \sec^2 \phi(1 - 0.2 \tan \phi - \tan^2 \phi)}{(0.1 + \tan \phi)^2} = 0$$

When $1 - 0.2 \tan \phi - \tan^2 \phi = 0$:

$$\tan \phi = \frac{-1 \pm \sqrt{101}}{10}$$

$$\phi \approx 42.1°, 132.1°.$$

By the First Derivative Test, the maximum occurs when $\phi \approx 42.1°$.

55. The slope of the line is $(-4/3)$, and the equation is

$$y - 0 = -\frac{4}{3}(x - 3)$$

$$y = -\frac{4}{3}x + 4$$

For the rectangle, let $(x, y) = (x, (-4/3)x + 4)$ be the vertex on the line. Then the area is

$$A = bh = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \Longrightarrow x = \frac{3}{2} \text{ and } y = 2. \text{ Base } = \frac{3}{2}, \text{ height } = 2.$$

For the circle, you can use geometry to note that the center of the inscribed circle is (r, r), and the distance from (r, r) to the line $(4/3)x + y - 4 = 0$ is also r:

$$\frac{|(4/3)r + r - 4|}{\sqrt{(16/9) + 1}} = r$$

$$|7r - 12| = 5r$$

$$7r - 12 = \pm 5r$$

$$r = 1, 6.$$

Clearly, we must choose $r = 1$. Hence, the center is $(1, 1)$ and the radius is 1.

For the semicircle, observe that the center must be of the form (r, r), and must lie on the line $y = (-4/3)x + 4$. Thus,

$$r = \frac{-4}{3}r + 4$$

$$3r = -4r + 12$$

$$7r = 12$$

$$r = \frac{12}{7}.$$

The center is $(12/7, 12/7)$ and the radius is $r = 12/7$.

Section 3.8 Newton's Method

1. $f(x) = x^2 - 3$

$f'(x) = 2x$

$x_1 = 1.7$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	−0.1100	3.4000	−0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

2. $f(x) = 3x^2 - 2$

$f'(x) = 6x$

$x_1 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	1.0000	6.0000	0.1667	0.8333
2	0.8333	0.0832	4.9998	0.0166	0.8166

3. $f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.1411	−0.9900	−0.1425	3.1425
2	3.1425	−0.0009	−1.0000	0.0009	3.1416

4. $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

5. $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

Approximation of the zero of f is 0.682.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.3750	1.7500	−0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

6. $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

Approximation of the zero of f is 0.755.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.4688	1.3125	−0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

7. $f(x) = 3\sqrt{x-1} - x$

$$f'(x) = \frac{3}{2\sqrt{x-1}} - 1$$

Approximation of the zero of f is 1.146.

Similarly, the other zero is approximately 7.854.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.2000	0.1416	2.3541	0.0602	1.1398
2	1.1398	−0.0181	3.0118	−0.0060	1.1458
3	1.1458	−0.0003	2.9284	−0.0001	1.1459

8. $f(x) = x^3 + 3$

$f'(x) = 3x^2$

Approximation of the zero of f is -1.442.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.5000	−0.3750	6.7500	−0.0556	−1.4444
2	−1.4444	−0.0134	6.2589	−0.0021	−1.4423
3	−1.4423	−0.0003	6.2407	−0.0001	−1.4422

9. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.3360	1.6400	−0.2049	0.7049
2	0.7049	−0.0921	0.7824	−0.1177	0.8226
3	0.8226	−0.0231	0.4037	−0.0573	0.8799
4	0.8799	−0.0045	0.2495	−0.0181	0.8980
5	0.8980	−0.0004	0.2048	−0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	−0.1600	−0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

10. $f(x) = x^4 - 10x^2 - 11$

$f'(x) = 4x^3 - 20x$

Approximation of the zero of f is 3.317.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.5000	16.5625	101.5000	0.1632	3.3368
2	3.3368	1.6288	81.8749	0.0199	3.3169
3	3.3169	0.0219	79.6298	0.0003	3.3166

11. $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of f is -0.489.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−0.5000	−0.0206	1.8776	−0.0110	−0.4890
2	−0.4890	0.0000	1.8723	0.0000	−0.4890

12. $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

Approximation of the zero of f is 0.866.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0001	3.0087	0.0000	0.8655

13. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x+4}$

$h'(x) = 2 - \dfrac{1}{2\sqrt{x+4}}$

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	−0.0001	1.7661	0.0000	0.5687

14. $h(x) = f(x) - g(x) = 3 - x - \dfrac{1}{x^2+1}$

$h'(x) = -1 + \dfrac{2x}{(x^2+1)^2}$

Point of intersection of the graphs of f and g occurs when $x \approx 2.893$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	2.9000	−0.0063	−0.9345	0.0067	2.8933
2	2.8933	0.0000	−0.9341	0.0000	2.8933

15. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of f and g occurs when $x \approx 4.493$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	4.5000	−0.1373	−21.5048	0.0064	4.4936
2	4.4936	−0.0039	−20.2271	0.0002	4.4934

16. $h(x) = f(x) - g(x) = x^2 - \cos x$

$h'(x) = 2x + \sin x$

One point of intersection of the graphs of f and g occurs when $x \approx 0.824$. Since $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the y-axis, the other point of intersection occurs when $x \approx -0.824$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.8000	−0.0567	2.3174	−0.0245	0.8245
2	0.8245	0.0009	2.3832	0.0004	0.8241

17. Let $g(x) = f(x) - x = \cos x - x$

$g'(x) = -\sin x - 1.$

The fixed point is approximately 0.74.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	−0.4597	−1.8415	0.2496	0.7504
2	0.7504	−0.0190	−1.6819	0.0113	0.7391
3	0.7391	0.0000	−1.6736	0.0000	0.7391

18. Let $g(x) = f(x) - x = \cot x - x$

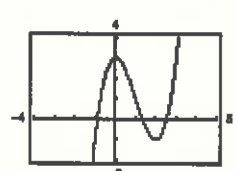
$g'(x) = -\csc^2 x - 1.$

The fixed point is approximately 0.86.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	−0.3579	−2.4123	0.1484	0.8516
2	0.8516	0.0240	−2.7668	−0.0087	0.8603
3	0.8603	0.0001	−2.7403	0.0000	0.8603

19. $f(x) = x^3 - 3x^2 + 3,\ f'(x) = 3x^2 - 6x$

(a)



—CONTINUED—

(b) $x_1 = 1$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 1.333$

Continuing, the zero is 1.347.

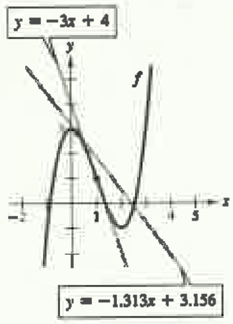
(c) $x_1 = \dfrac{1}{4}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 2.405$

Continuing, the zero is 2.532.

19. —CONTINUED—

(d)

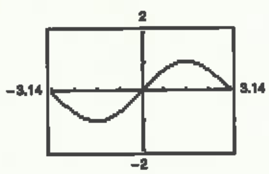


The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

20. $f(x) = \sin x$, $f'(x) = \cos x$

(a)



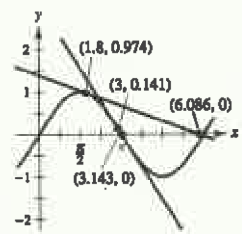
(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

21. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

22. $y = 4x^3 - 12x^2 + 12x - 3 = f(x)$

$y' = 12x^2 - 24x + 12 = f'(x)$

$x_1 = \frac{3}{2}$

$f'(x_2) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	$\frac{3}{2}$	$\frac{3}{2}$	3	$\frac{1}{2}$	1
2	1	1	0	—	—

23. $y = -x^3 + 3x^2 - x + 1 = f(x)$

$y' = -3x^2 + 6x - 1 = f'(x)$

$x_1 = 1$

Fails to converge.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	2	2	1	0
2	0	1	−1	−1	1
3	1	2	2	1	0

24. $f(x) = 2\sin x + \cos 2x$

$f'(x) = 2\cos x - 2\sin 2x$

$x_1 = \dfrac{3\pi}{2}$

Fails because $f'(x_1) = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
1	$\frac{3\pi}{2}$	-3	0

25. $f(x) = x^2 - a = 0$

$f'(x) = 2x$

$$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i}$$

$$= \frac{2x_i^2 - x_i^2 + a}{2x_i} = \frac{x_i^2 + a}{2x_i} = \frac{x_i}{2} + \frac{a}{2x_i}$$

26. $f(x) = x^n - a = 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{x_i^n - a}{nx_i^{n-1}}$$

$$= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

27. $x_{i+1} = \dfrac{x_i^2 + 7}{2x_i}$

i	1	2	3	4	5
x_i	2.0000	2.7500	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

28. $x_{i+1} = \dfrac{x_i^2 + 5}{2x_i}$

i	1	2	3	4
x_i	2.0000	2.2500	2.2361	2.2361

$\sqrt{5} \approx 2.236$

29. $x_{i+1} = \dfrac{3x_i^4 + 6}{4x_i^3}$

i	1	2	3	4
x_i	1.5000	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

30. $x_{i+1} = \dfrac{2x_i^3 + 15}{3x_i^2}$

i	1	2	3	4
x_i	2.5000	2.4667	2.4662	2.4662

$\sqrt[3]{15} \approx 2.466$

31. $f(x) = \dfrac{1}{x} - a = 0$

$f'(x) = -\dfrac{1}{x^2}$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

32. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$\frac{1}{3} \approx 0.333$

(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$\frac{1}{11} \approx 0.091$

33. $f(x) = 1 + \cos x$

$f'(x) = -\sin x$

Approximation of the zero: 3.141

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.0100	−0.1411	−0.0709	3.0709
2	3.0709	0.0025	−0.0706	−0.0354	3.1063
3	3.1063	0.0006	−0.0353	−0.0176	3.1239
4	3.1239	0.0002	−0.0177	−0.0088	3.1327
5	3.1327	0.0000	−0.0089	−0.0044	3.1371
6	3.1371	0.0000	−0.0045	−0.0022	3.1393
7	3.1393	0.0000	−0.0023	−0.0011	3.1404
8	3.1404	0.0000	−0.0012	−0.0006	3.1410

34. $f(x) = \tan x$

$f'(x) = \sec^2 x$

Approximation of the zero: 3.142

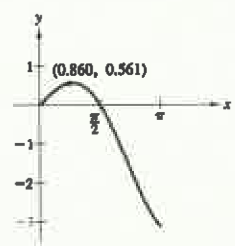
n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	−0.1425	1.0203	−0.1397	3.1397
2	3.1397	−0.0019	1.0000	−0.0019	3.1416
3	3.1416	0.0000	1.0000	0.0000	3.1416

35. $f(x) = x \cos x$

$f'(x) = -x \sin x + \cos x = 0$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$[F'(x) = -2 \sin x + x \cos x]$



n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	0.9000	−0.0834	−2.1261	0.0392	0.8608
2	0.8608	−0.0010	−2.0778	0.0005	0.8603

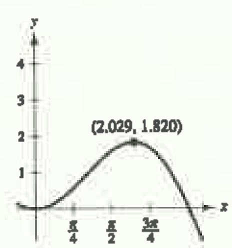
Approximation to the critical number: 0.860

36. $f(x) = x \sin x$

$f'(x) = x \cos x + \sin x = 0$

One root is $x = 0$. Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$[F'(x) = 2 \cos x - x \sin x]$



n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	2.0000	0.0770	−2.6509	−0.0290	2.0290
2	2.0290	−0.0007	−2.7044	0.0002	2.0288

Approximation to the critical number: 2.029

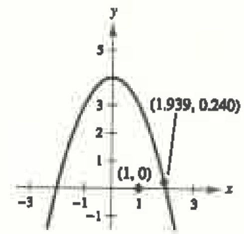
37. $y = f(x) = 4 - x^2, (1, 0)$

$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$

d is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

$g(x) = D' = 4x^3 - 14x - 2$

$g'(x) = 12x^2 - 14$



n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	−0.0012	31.0934	0.0000	1.9385

$x \approx 1.939$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.

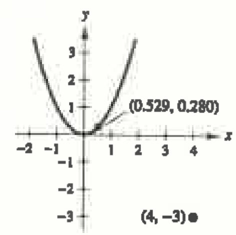
38. $y = f(x) = x^2, (4, -3)$

$d = \sqrt{(x-4)^2 + (y+3)^2} = \sqrt{(x-4)^2 + (x^2+3)^2} = \sqrt{x^4 + 7x^2 - 8x + 25}$

d is minimum when $D = x^4 + 7x^2 - 8x + 25$ is minimum.

$g(x) = D' = 4x^3 + 14x - 8$

$g'(x) = 12x^2 + 14$



n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	0.5000	−0.5000	17.0000	−0.0294	0.5294
2	0.5294	0.0051	17.3632	0.0003	0.5291
3	0.5291	−0.0001	17.3594	0.0000	0.5291

$x \approx 0.529$

Point closest to $(4, -3)$ is approximately $(0.529, 0.280)$.

39.

$$\text{Minimize: } T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$$

$$T = \frac{\sqrt{x^2+4}}{3} + \frac{\sqrt{x^2-6x+10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}} = 0$$

$$4x\sqrt{x^2-6x+10} = -3(x-3)\sqrt{x^2+4}$$

$$16x^2(x^2-6x+10) = 9(x-3)^2(x^2+4)$$

$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Since $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	−1.0480	150.2780	−0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ miles

40. Maximize: $C = \dfrac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300.$

Since $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

41. $$2{,}500{,}000 = -76x^3 + 4830x^2 - 320{,}000$$

$76x^3 - 4830x^2 + 2{,}820{,}000 = 0$

Let $f(x) = 76x^3 - 4830x^2 + 2{,}820{,}000$

$f'(x) = 228x^2 - 9660x.$

From the graph, choose $x_1 = 40$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	40.0000	−44000.0000	−21600.0000	2.0370	37.9630
2	37.9630	17157.6209	−38131.4039	−0.4500	38.4130
3	38.4130	780.0914	−34642.2263	−0.0225	38.4355
4	38.4355	2.6308	−34465.3435	−0.0001	38.4356

The zero occurs when $x \approx 38.4356$ which corresponds to \$384,356.

42. $170 = 0.808x^3 - 17.974x^2 + 71.248x + 110.843, 1 \le x \le 5$

Let $f(x) = 0.808x^3 - 17.974x^2 + 71.248x - 59.157$

$f'(x) = 2.424x^2 - 35.948x + 71.248.$

From the graph, choose $x_1 = 1$ and $x_1 = 3.5$. Apply Newton's Method.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	−5.0750	37.7240	−0.1345	1.1345
2	1.1345	−0.2805	33.5849	−0.0084	1.1429
3	1.1429	0.0006	33.3293	0.0000	1.1429

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.5000	4.6725	−24.8760	−0.1878	3.6878
2	3.6878	−0.3286	−28.3550	0.0116	3.6762
3	3.6762	−0.0009	−28.1450	0.0000	3.6762

The zeros occur when $x \approx 1.1429$ and $x \approx 3.6762$. These approximately correspond to engine speeds of 1143 rev/min and 3676 rev/min.

43. False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ is given in reduced form.

44. True

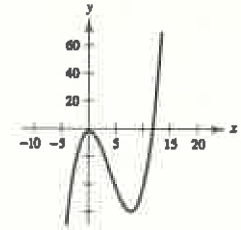
45. True

46. True

47. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

$f'(x) = \frac{3}{4}x^2 - 6x + \frac{3}{4}$

Let $x_1 = 12$.



n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	12.0000	7.0000	36.7500	0.1905	11.8095
2	11.8095	0.2151	34.4912	0.0062	11.8033
3	11.8033	0.0015	34.4186	0.0000	11.8033

Approximation: $x \approx 11.803$

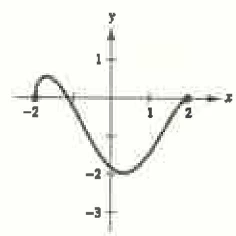
48. $f(x) = \sqrt{4 - x^2}\sin(x - 2)$

Domain: $[-2, 2]$

$x = -2$ and $x = 2$ are both zeros.

$$f'(x) = \sqrt{4 - x^2}\cos(x - 2) - \frac{x}{\sqrt{4 - x^2}}\sin(x - 2)$$

Let $x_1 = -1$.



n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.0000	−0.2444	−1.7962	0.1361	−1.1361
2	−1.1361	−0.0090	−1.6498	0.0055	−1.1416
3	−1.1416	0.0000	−1.6422	0.0000	−1.1416

Zeros: $x = \pm 2, x \approx -1.142$

Section 3.9 Differentials

1. $f(x) = x^2$

$f'(x) = 2x$

Tangent line at $(2, 4)$: $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

2. $f(x) = \frac{1}{x^2}$

$f'(x) = -\frac{2}{x^3}$

Tangent line at $\left(2, \frac{1}{4}\right)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \frac{1}{4} = -\frac{1}{4}(x - 2)$$

$$y = -\frac{1}{4}x + \frac{3}{4}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \frac{1}{x^2}$	0.2770	0.2525	0.2500	0.2475	0.2268
$T(x) = -\frac{1}{4}x + \frac{3}{4}$	0.2750	0.2525	0.2500	0.2475	0.2250

3. $f(x) = x^5$

$f'(x) = 5x^4$

Tangent line at (2, 32): $y - f(2) = f'(2)(x - 2)$

$y - 32 = 80(x - 2)$

$y = 80x - 128$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

4. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $(2, \sqrt{2})$:

$y - f(2) = f'(2)(x - 2)$

$y - \sqrt{2} = \dfrac{1}{2\sqrt{2}}(x - 2)$

$y = \dfrac{x}{2\sqrt{2}} + \dfrac{1}{\sqrt{2}}$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \dfrac{x}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

5. $f(x) = \sin x$

$f'(x) = \cos x$

Tangent line at $(2, \sin 2)$:

$y - f(2) = f'(2)(x - 2)$

$y - \sin 2 = (\cos 2)(x - 2)$

$y = (\cos 2)(x - 2) + \sin 2$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

6. $f(x) = \csc x$

$f'(x) = -\csc x \cot x$

Tangent line at $(2, \csc 2)$: $y - f(2) = f'(2)(x - 2)$

$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$

$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$

x	1.9	1.99	2	2.01	2.1
$f(x) = \csc x$	1.0567	1.0948	1.0998	1.1049	1.1585
$T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$	1.0494	1.0947	1.0998	1.1048	1.1501

7. $y = f(x) = x^3, f'(x) = 3x^2, x = 1, \Delta x = dx = 0.1$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= f(1.1) - f(1) \\ &= (1.1)^3 - (1)^3 \\ &= 0.331\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx \\ &= f'(1)(0.1) \\ &= 3(0.1) \\ &= 0.3\end{aligned}$$

8. $y = f(x) = 1 - 2x^2, f'(x) = -4x, x = 0, \Delta x = dx = -0.1$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) \\ &= f(-0.1) - f(0) \\ &= [1 - 2(-0.1)^2] - [1 - 2(0)^2] = -0.02\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx \\ &= f'(0)(-0.1) \\ &= (0)(-0.1) = 0\end{aligned}$$

9. $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x) = f(-0.99) - f(-1) = [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394$$

$$dy = f'(x)\,dx = f'(-1)(0.01) = (-4)(0.01) = -0.04$$

10. $y = f(x) = 2x + 1, f'(x) = 2, x = 2, \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x) = f(2.01) - f(2) = [2(2.01) + 1] - [2(2) + 1] = 0.02$$

$$dy = f'(x)\,dx = f'(2)(0.01) = 2(0.01) = 0.02$$

11. $y = 3x^2 - 4$

$dy = 6x\,dx$

12. $y = 2x^{3/2}$

$dy = 3\sqrt{x}\,dx$

13. $y = \dfrac{x+1}{2x-1}$

$dy = \dfrac{-3}{(2x-1)^2}\,dx$

14. $y = \sqrt{x^2 - 4}$

$dy = \dfrac{x}{\sqrt{x^2-4}}\,dx$

15. $y = x\sqrt{1 - x^2}$

$$dy = \left(\frac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right)dx = \frac{1 - 2x^2}{\sqrt{1-x^2}}\,dx$$

16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$$dy = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)dx = \frac{x-1}{2x\sqrt{x}}\,dx$$

17. $y = \dfrac{\sec^2 x}{x^2 + 1}$

$$dy = \left[\frac{(x^2+1)2\sec^2 x\tan x - \sec^2 x(2x)}{(x^2+1)^2}\right]dx = \left[\frac{2\sec^2 x(x^2\tan x + \tan x - x)}{(x^2+1)^2}\right]dx$$

18. $y = x\sin x$

$dy = (x\cos x + \sin x)\,dx$

19. $y = \dfrac{1}{3}\cos\left(\dfrac{6\pi x - 1}{2}\right)$

$$dy = -\pi\sin\left(\frac{6\pi x - 1}{2}\right)dx$$

20. $y = x - \tan^2 x$

$dy = (1 - 2\tan x\sec^2 x)\,dx$

21. $A = x^2$

$x = 12$

$\Delta x = dx = \pm\frac{1}{64}$

$dA = 2x\,dx$

$\Delta A \approx dA = 2(12)\left(\pm\frac{1}{64}\right)$

$= \pm\frac{3}{8}$ square inches

22. $A = \frac{1}{2}bh, b = 36, h = 50$

$db = dh = \pm 0.25$

$dA = \frac{1}{2}b\,dh + \frac{1}{2}h\,db$

$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$

$= \pm 10.75$ square centimeters

23. $A = \pi r^2$

$r = 14$

$\Delta r = dr = \pm\frac{1}{4}$

$\Delta A \approx dA = 2\pi r\,dr = \pi(28)\left(\pm\frac{1}{4}\right)$

$= \pm 7\pi$ square inches

24. $x = 12$ inches

$\Delta x = dx = \pm 0.03$ inch

(a) $V = x^3$

$dV = 3x^2\,dx = 3(12)^2(\pm 0.03)$

$= \pm 12.96$ cubic inches

(b) $S = 6x^2$

$dS = 12x\,dx = 12(12)(\pm 0.03)$

$= \pm 4.32$ square inches

25. (a) $x = 15$ centimeter

$\Delta x = dx = \pm 0.05$ centimeters

$A = x^2$

$dA = 2x\,dx = 2(15)(\pm 0.05)$

$= \pm 1.5$ square centimeters

Percentage error:

$$\frac{dA}{A} = \frac{\pm 1.5}{(15)^2} = 0.00666\ldots = \frac{2}{3}\%$$

(b) $\dfrac{dA}{A} = \dfrac{2x\,dx}{x^2} = \dfrac{2\,dx}{x} \le 0.025$

$$\frac{dx}{x} \le \frac{0.025}{2} = 0.0125 = 1.25\%$$

26. (a) $C = 56$ centimeters

$\Delta C = dC = \pm 1.2$ centimeters

$$C = 2\pi r \Longrightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{1}{4\pi}C^2$$

$$dA = \frac{1}{2\pi}C\,dC = \frac{1}{2\pi}(56)(\pm 1.2) = \frac{33.6}{\pi}$$

$$\frac{dA}{A} = \frac{33.6/\pi}{[1/(4\pi)](56)^2} \approx 0.042857 = 4.2857\%$$

(b) $\dfrac{dA}{A} = \dfrac{(1/2\pi)C\,dC}{(1/4\pi)C^2} = \dfrac{2dC}{C} \le 0.03$

$$\frac{dC}{C} \le \frac{0.03}{2} = 0.015 = 1.5\%$$

27. $r = 6$ inches

$\Delta r = dr = \pm 0.02$ inches

(a) $V = \dfrac{4}{3}\pi r^3$

$dV = 4\pi r^2\,dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi$ cubic inches

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi$ square inches

(c) Relative error: $\dfrac{dV}{V} = \dfrac{4\pi r^2\,dr}{(4/3)\pi r^3} = \dfrac{3dr}{r}$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

Relative error: $\dfrac{dS}{S} = \dfrac{8\pi r\,dr}{4\pi r^2} = \dfrac{2dr}{r}$

$$= \frac{2(0.02)}{6} = 0.000666\ldots = \frac{2}{3}\%$$

28. $P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$, x changes from 115 to 120

$dP = (500 - 2x - x + 77)dx = (577 - 3x)\,dx = [577 - 3(115)](120 - 115) = 1160$

Approximate percentage change: $\dfrac{dP}{P}(100) = \dfrac{1160}{43517.50}(100) \approx 2.7\%$

29. $V = \pi r^2 h = 40\pi r^2$, $r = 5$ cm, $h = 40$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 80\pi r\,dr = 80\pi(5)(0.2) = 80\pi$ cm^3

30. $V = \frac{4}{3}\pi r^3$, $r = 100$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 4\pi r^2\,dr = 4\pi(100)^2(0.2) = 8000\pi$ cm^3

31. (a) $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}}\,dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi\,dL)/\left(g\sqrt{L/g}\right)}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2}\text{ (relative error in } L\text{)}$$

$$= \frac{1}{2}(0.005) = 0.0025$$

Percentage error: $\frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$

(b) $(0.0025)(3600)(24) = 216$ seconds

$= 3.6$ minutes

32. $T = 2\pi\sqrt{L/g}$

$$T^2 = 4\pi^2\left(\frac{L}{g}\right)$$

$$g = \frac{4\pi^2 L}{T^2}$$

$$dg = 4\pi^2 L\frac{(-2)}{T^3}\,dT$$

Relative error in g:

$$\frac{dg}{g} = \frac{(-8\pi^2 L\,dT)/T^3}{(4\pi^2 L)/T^2}$$

$$= -2\frac{dT}{T} = -2\text{ (relative error in period)}$$

$$= -2(0.001) = -0.002$$

Percentage error in g: $\left|\frac{dg}{g}\right|(100) = \frac{1}{5}\%$

33. $E = IR$

$$R = \frac{E}{I}$$

$$dR = -\frac{E}{I^2}\,dI$$

$$\frac{dR}{R} = \frac{-(E/I^2)\,dI}{E/I} = -\frac{dI}{I}$$

$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

34. $C = \frac{80{,}000p}{100 - p},\ 0 \le p < 100$

(a) $p = 40$

$\Delta p = dp = 2$

$$dC = \frac{8{,}000{,}000}{(100-p)^2}\,dp$$

$$= \frac{8{,}000{,}000}{(100-40)^2}(2) \approx \$4444.44$$

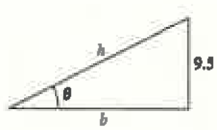
(b) $p = 75$

$\Delta p = dp = 2$

$$dC = \frac{8{,}000{,}000}{(100-75)^2}(2) = \$25{,}600.00$$

35. $\theta = 26°45' = 26.75°$

$d\theta = \pm 15' = \pm 0.25°$



(a) $h = 9.5\csc\theta$

$$dh = -9.5\csc\theta\cot\theta\,d\theta$$

$$\frac{dh}{h} = -\cot\theta\,d\theta$$

$$\left|\frac{dh}{h}\right| = (\cot 26.75°)(0.25°)$$

Converting to radians, $(\cot 0.4669)(0.0044)$
$\approx 0.0087 = 0.87\%$ (in radians).

(b) $\left|\frac{dh}{h}\right| = \cot\theta\,d\theta \le 0.02$

$$\frac{d\theta}{\theta} \le \frac{0.02}{\theta(\cot\theta)} = \frac{0.02\tan\theta}{\theta}$$

$$\frac{d\theta}{\theta} \le \frac{0.02\tan 26.75°}{26.75°} \approx \frac{0.02\tan 0.4669}{0.4669}$$

$\approx 0.0216 = 2.16\%$ (in radians)

36. See Exercise 35.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(9.5\cot\theta)(9.5) = 45.125\cot\theta$$

$$dA = -45.125\csc^2\theta\,d\theta$$

$$\left|\frac{dA}{A}\right| = \frac{\csc^2\theta\,d\theta}{\cot\theta} = \frac{d\theta}{\sin\theta\cos\theta}$$

$$= \frac{0.25°}{(\sin 26.75°)(\cos 26.75°)}$$

$$\approx \frac{0.0044}{(\sin 0.4669)(\cos 0.4669)}$$

$$\approx 0.0109 = 1.09\%\ (\text{in radians})$$

37. $r = \dfrac{v_0^2}{32}(\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$$dr = \frac{(2200)^2}{16}(\cos 2\theta)\,d\theta$$

$$\theta = 10\left(\frac{\pi}{180}\right)$$

$$d\theta = (11-10)\frac{\pi}{180}$$

$$\Delta r \approx dr$$

$$= \frac{(2200)^2}{16}\cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right) \approx 4961 \text{ feet}$$

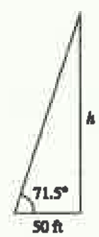
38. $h = 50\tan\theta$

$\theta = 71.5°$ (see figure)

$$dh = 50\sec^2\theta\,d\theta$$

$$\frac{dh}{h} = \frac{\sec^2 71.5°}{\tan 71.5°}\,d\theta \le \pm 0.06$$

$$d\theta \le \pm 0.018 \text{ radians}$$



39. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$$f(x+\Delta x) \approx f(x) + f'(x)\,dx$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$$

$$f(x+\Delta x) = \sqrt{99.4}$$

$$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

40. Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = 1$.

$$f(x+\Delta x) \approx f(x) + f'(x)\,dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}\,dx$$

$$f(x+\Delta x) = \sqrt[3]{28} \approx \sqrt[3]{27} + \frac{1}{3(\sqrt[3]{27})^2}(1) = 3 + \frac{1}{27} \approx 3.0370$$

Using a calculator: $\sqrt[3]{28} \approx 3.0366$

41. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$$f(x+\Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}\,dx$$

$$f(x+\Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4(\sqrt[4]{625})^3}(-1)$$

$$= 5 - \frac{1}{500} = 4.998$$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

42. Let $f(x) = x^3, x = 3, dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2\,dx$$
$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01) = 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

43. Let $f(x) = x^4, x = 1, dx = -0.01, f'(x) = 4x^3$.

Then

$$f(0.99) \approx f(1) + f'(1)dx$$
$$(0.99)^4 \approx (1)^4 + 4(1)^3(-0.01) = 1 - 4(0.01).$$

44. Let $f(x) = \sqrt{x}, x = 4, dx = 0.02, f'(x) = 1/(2\sqrt{x})$.

Then

$$f(4.02) \approx f(4) + f'(4)dx$$
$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$$

45. Let $f(x) = \sec x, x = 0, dx = 0.03, f'(x) = \sec x \tan x$.

Then

$$f(0.03) \approx f(0) + f'(0)\,dx$$
$$\sec 0.03 \approx \sec 0 + (\sec 0 \tan 0)(0.03) = 1 + 0(0.03).$$

46. Let $f(x) = \tan x, x = 0, dx = 0.05, f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$
$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

47. True

48. True, $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = a$

49. True

50. False

Let $f(x) = \sqrt{x}, x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

$$dy = f'(x)\,dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

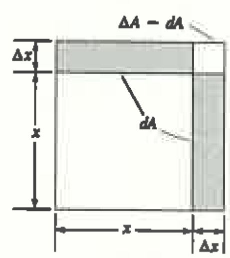
Thus, $dy > \Delta y$ in this example.

51. $A(x) = x^2$

(a) $dA = 2x\,dx = 2x\,\Delta x$

$$\Delta A = (x + \Delta x)^2 - x^2 = 2x\,\Delta x + (\Delta x)^2$$

(b)

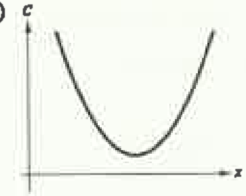


(c) $\Delta A - dA = (\Delta x)^2$

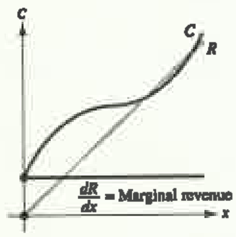
52. Let $\varepsilon = (\Delta y/\Delta x) - f'(x)$. Then $\Delta x\varepsilon = \Delta y - f'(x)\,\Delta x$. Since $\Delta x = dx$, we have $\Delta y - f'(x)\,dx = \Delta x\varepsilon$ or $\Delta y - dy = \varepsilon\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$. Thus, $\Delta y - dy$ tends to 0 very rapidly, since $\Delta x \to 0$ and $\varepsilon \to 0$.

Section 3.10 Business and Economics Applications

1. (a) $C(0)$ represents the fixed costs.

(b) 

(c) The marginal cost function has a relative minimum. It occurs when production costs are increasing at their slowest rate.

2. (a) 

(b) 

3. $R = 900x - 0.1x^2$

$$\frac{dR}{dx} = 900 - 0.2x = 0 \text{ when } x = 4500.$$

By the First Derivative Test, $x = 4500$ is a maximum.

4. $R = 600x^2 - 0.02x^3$

$$\frac{dR}{dx} = 1200x - 0.06x^2$$

$$= 6x(200 - 0.01x) = 0 \text{ when } x = 0, 20{,}000.$$

By the First Derivative Test, $x = 20{,}000$ is a maximum.

5. $$R = \frac{1{,}000{,}000x}{0.02x^2 + 1800}$$

$$\frac{dR}{dx} = 1{,}000{,}000\left[\frac{0.02x^2 + 1800 - x(0.04x)}{(0.02x^2 + 1800)^2}\right] = 0$$

$1800 - 0.02x^2 = 0$ when $x = 300$.

By the First Derivative Test, $x = 300$ is a maximum.

6. $R = 30x^{2/3} - 2x$

$$\frac{dR}{dx} = 20^{-1/3} - 2 = \frac{20}{x^{1/3}} - 2 = 0 \text{ when } x = 1000.$$

By the First Derivative Test, $x = 1000$ is a maximum.

7. $\overline{C} = 0.125x + 20 + \dfrac{5000}{x}$

$$\frac{d\overline{C}}{dx} = 0.125 - \frac{5000}{x^2} = 0 \text{ when } x = 200.$$

By the First Derivative Test, $x = 200$ yields the minimum average cost.

8. $\overline{C} = 0.001x^2 - 5 + \dfrac{250}{x}$

$$\frac{d\overline{C}}{dx} = 0.002x - \frac{250}{x^2} = 0 \text{ when } x = 50.$$

By the First Derivative Test, $x = 50$ yields the minimum average cost.

9. $\overline{C} = 3000 - x(300 - x)^{1/2}$

$$\frac{d\overline{C}}{dx} = -x\left(\frac{1}{2}\right)(300 - x)^{-1/2}(-1) - (300 - x)^{1/2} = -\frac{3}{2}(300 - x)^{-1/2}(200 - x) = 0 \text{ when } x = 200.$$

By the First Derivative Test, $x = 200$ yields the minimum average cost.

10. $\overline{C} = \dfrac{2x^2 - x + 5000}{x^2 + 2500}$

$$\frac{d\overline{C}}{dx} = \frac{(x^2 + 2500)(4x - 1) - (2x^2 - x + 5000)(2x)}{(x^2 + 2500)^2} = \frac{x^2 - 2500}{(x^2 + 2500)^2} = 0 \text{ when } x = 50.$$

By the First Derivative Test, $x = 50$ yields the minimum average cost.

11. $C = 100 + 30x$

$p = 90 - x$

$P = xp - C$

$= 90x - x^2 - 30x - 100$

$= -x^2 + 60x - 100$

$\dfrac{dP}{dx} = -2x + 60$

$= 0$ when $x = 30$, so $p = 60$.

By the First Derivative Test, $x = 30$ is a maximum.

12. $C = 2400x + 5200$

$p = 6000 - 0.4x^2$

$P = xp - C$

$= (6000x - 0.4x^3) - (2400x + 5200)$

$= -0.4x^3 + 3600x - 5200$

$\dfrac{dP}{dx} = -1.2x^2 + 3600$

$= 0$ when $x \approx 55$, so $p \approx 4800$.

By the First Derivative Test, $x \approx 55$ is a maximum.

13. $C = 4000 - 40x + 0.02x^2$

$p = 50 - 0.01x$

$P = xp - C = 50x - 0.01x^2 - 4000 + 40x - 0.02x^2 = -0.03x^2 + 90x - 4000$

$\dfrac{dP}{dx} = -0.06x + 90 = 0$ when $x = 1500$, so $p = 35$.

By the First Derivative Test, $x = 1500$ is a maximum.

14. $C = 35x + 2\sqrt{x - 1}$

$p = 40 - \sqrt{x - 1}$

$P = xp - C = 40x - x\sqrt{x - 1} - 35x - 2\sqrt{x - 1} = 5x - (x + 2)\sqrt{x - 1}$

$$\frac{dP}{dx} = 5 - \left[(x + 2)\left(\frac{1}{2}\right)(x - 1)^{-1/2} + (x - 1)^{1/2}\right]$$

$$= 5 - \left[\frac{x + 2 + 2(x - 1)}{2\sqrt{x - 1}}\right] = 5 - \frac{3x}{2\sqrt{x - 1}} = \frac{10\sqrt{x - 1} - 3x}{2\sqrt{x - 1}} = 0 \text{ when } x = 10, \text{ so } p = 37.$$

By the First Derivative Test, $x = 10$ is a maximum.

15. $C = 2x^2 + 5x + 18$

Average cost $= \dfrac{C}{x} = \overline{C} = 2x + 5 + \dfrac{18}{x}$

$\dfrac{d\overline{C}}{dx} = 2 - \dfrac{18}{x^2} = 0$ when $x = 3$.

$\overline{C}(3) = 6 + 5 + 6 = 17$

By the First Derivative Test, $x = 3$ is a minimum.

Marginal cost: $\dfrac{dC}{dx} = 4x + 5$

At $x = 3$: $\dfrac{dC}{dx} = 17 = \overline{C}(3)$

16. $C = x^3 - 6x^2 + 13x$

Average cost $= \dfrac{C}{x} = \overline{C} = x^2 - 6x + 13$

$\dfrac{d\overline{C}}{dx} = 2x - 6 = 0$ when $x = 3$.

$\dfrac{dC}{dx} = 3x^2 - 12x + 13$ when $x = 3$.

Marginal cost $= \dfrac{dC}{dx} = 27 - 36 + 13 = 4$

Average cost $= \overline{C} = 9 - 18 + 13 = 4$

17. Average cost: $\overline{C}(x) = \dfrac{C(x)}{x}$

$$\frac{d\overline{C}}{dx} = \frac{xC'(x) - C(x)}{x^2} = 0 \Longrightarrow xC'(x) - C(x) = 0 \text{ when } C'(x) = \frac{C(x)}{x} = \overline{C}(x).$$

Marginal cost = average cost

This condition will yield a minimum (if it exists).

18. $P = 230 + 20s - \dfrac{1}{2}s^2$

$\dfrac{dP}{ds} = 20 - s = 0$ when $s = 20$.

P is maximum when advertising is \$2000 since $d^2P/ds^2 = -1 < 0$.

19. (a)

Order size, x	Price	Profit
102	$90 - 2(0.15)$	$102[90 - 2(0.15)] - 102(60) = 3029.40$
104	$90 - 4(0.15)$	$104[90 - 4(0.15)] - 104(60) = 3057.60$
106	$90 - 6(0.15)$	$106[90 - 6(0.15)] - 106(60) = 3084.60$
108	$90 - 8(0.15)$	$108[90 - 8(0.15)] - 108(60) = 3110.40$
110	$90 - 10(0.15)$	$110[90 - 10(0.15)] - 110(60) = 3135.00$
112	$90 - 12(0.15)$	$112[90 - 12(0.15)] - 112(60) = 3158.40$

(b)

Order size, x	Profit
148	3374.40
149	3374.90
150	3375.00
151	3374.90
152	3374.40

The maximum profit is 3375.00 for $x = 150$.

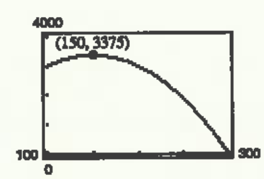
(c) $P(x) = x[90 - (x - 100)(0.15)] - 60x$

$= x(45 - 0.15x), x \geq 100$

(d) $\dfrac{dP}{dx} = 45 - 0.30x = 0$ when $x = 150$.

Since $\dfrac{d^2P}{dx^2} < 0$, an order size of $x = 150$ units yields a maximum profit.

(e)



20. Let x be the number of \$40 increases in rent.

Profit $= P(x) = (50 - x)(720 + 40x) - (50 - x)(48)$

$= 33{,}600 + 1328x - 40x^2$

$P'(x) = 1328 - 80x = 0$ when $x = 16.6 \approx 17$.

Since $P''(x) < 0$, this yields a maximum. The rent should be $720 + 40(17) = \$1400$.

21. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33{,}000}{60v^2}$$

$= 0$ when $v = \sqrt{3000} = 10\sqrt{30} \approx 54.8$ mph.

$\dfrac{d^2T}{dv^2} = \dfrac{1100}{v^3} > 0$ when $v = 10\sqrt{30}$ so this value yields a minimum.

22. Total cost $=$ (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\frac{dT}{dv} = \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41{,}250}{50v^2}$$

$$= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

23. The total cost $T(x)$ is

$$T(x) = 12(5280)(6 - x) + 16(5280)\sqrt{x^2 + \frac{1}{4}}$$

$$T'(x) = 5280\left[-12 + \frac{16x}{\sqrt{x^2 + (1/4)}}\right] = 0$$

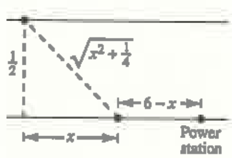
$$12 = \frac{16x}{\sqrt{x^2 + (1/4)}}$$

$$3\sqrt{x^2 + (1/4)} = 4x$$

$$9\left(x^2 + \frac{1}{4}\right) = 16x^2$$

$$\frac{9}{4} = 7x^2 \Longrightarrow x = \frac{3}{2\sqrt{7}} \approx 0.57 \text{ miles.}$$

This is a minimum by the First Derivative Test.



24. Let k be the cost per mile for laying pipe on land, and $T(x)$ the total cost.

$$T = 2k\sqrt{x^2 + 4} + k(4 - x)$$

$$T'(x) = k\left[2\left(\frac{x}{\sqrt{x^2 + 4}}\right) - 1\right] = 0$$

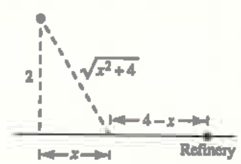
$$\frac{2x}{\sqrt{x^2 + 4}} = 1$$

$$2x = \sqrt{x^2 + 4}$$

$$4x^2 = x^2 + 4$$

$$3x^2 = 4 \Longrightarrow x = \frac{2}{\sqrt{3}} \approx 1.15 \text{ miles}$$

By the First Derivative Test, this is a minimum.



25. $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}.$$

Line: $y = \frac{64}{141}x$

$$S = \left|4\left(\frac{64}{141}\right) - 1\right| + \left|5\left(\frac{64}{141}\right) - 6\right| + \left|10\left(\frac{64}{141}\right) - 3\right|$$

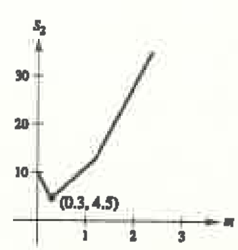
$$= \left|\frac{256}{141} - 1\right| + \left|\frac{320}{141} - 6\right| + \left|\frac{640}{141} - 3\right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

26. $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7 \text{ mi.}$$



27. $S_3 = \dfrac{|4m - 1|}{\sqrt{m^2 + 1}} + \dfrac{|5m - 6|}{\sqrt{m^2 + 1}} + \dfrac{|10m - 3|}{\sqrt{m^2 + 1}}$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$

28. By the methods of calculus, finding the minimum of the sums of the squares of the lengths of the vertical feeder lines is easiest (see Exercise 25). The minimum of the sum of the perpendicular distances best meets the objective of minimizing the amount of feeder line required (see Exercise 27). This minimum is easily found by using a graphing utility.

29. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$P = (0.12)d - id$

$d = ki^2$ (since d is proportional to i^2)

$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$

$\dfrac{dP}{di} = k(0.24i - 3i^2) = 0$ when $i = \dfrac{0.24}{3} = 0.08$.

$\dfrac{d^2P}{di^2} = k(0.24 - 6i) < 0$ when $i = 0.08$ (Note: $k > 0$).

The profit is a maximum when $i = 8\%$.

30. For the linear demand function, the rate of change (slope) is $m = -\frac{25}{5}$. Therefore, the demand x is

$x = 800 - \dfrac{25}{5}(p - 25) = 925 - 5p$

$R = xp = -5p^2 + 925p$

$\dfrac{dR}{dp} = -10p + 925 = 0$ when $p = \$92.50$.

31. $C = 100\left(\dfrac{200}{x^2} + \dfrac{x}{x + 30}\right),\ 1 \le x$

$C' = 100\left(-\dfrac{400}{x^3} + \dfrac{30}{(x + 30)^2}\right)$

$\overline{C}' = f(x) = 100\left(-\dfrac{400}{x^3} + \dfrac{30}{(x + 30)^2}\right)$

$f'(x) = 100\left(\dfrac{1200}{x^4} - \dfrac{60}{(x + 30)^3}\right)$

Approximation: $x \approx 40$ units

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	40.0000	−0.0128	0.0294	−0.4341	40.4341
2	40.4341	−0.0004	0.0277	−0.0131	40.4472
3	40.4472	0.0000	0.0277	0.0000	40.4472

32. $\overline{C} = \dfrac{C}{x} = \dfrac{800}{x} + 0.4 + 0.02x + 0.0001x^2$

$\overline{C}' = -\dfrac{800}{x^2} + 0.02 + 0.0002x$

$\overline{C}' = f(x) = -\dfrac{800}{x^2} + 0.02 + 0.0002x$

$f'(x) = \dfrac{1600}{x^3} + 0.0002$

Approximation: $x \approx 131$ units

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	130.0000	−0.0013	0.0009	−1.4406	131.4406
2	131.4406	0.0000	0.0009	0.0000	131.4406

33. $R = 900x - 0.1x^2$

$x = 3000$

$dx = 100$

$$dR = (900 - 0.2x)dx$$
$$= [900 - 0.2(3000)](100)$$
$$= \$30{,}000$$

34. $P = (500 - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$

$x = 175$

$dx = 5$

$$dP = [(500 - 2x) - (x - 77)]dx$$
$$= (-3x + 577)dx$$
$$= [-3(175) + 577](5)$$
$$= \$260$$

$$\frac{dP}{P} = \frac{260}{52{,}037.5} \approx 0.004996 \approx 0.5\% \text{ or } \frac{1}{2}\%$$

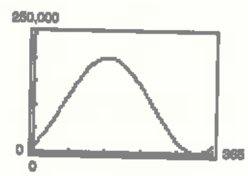
35. $F = 100{,}000\left(1 + \sin\left[\frac{2\pi(t - 60)}{365}\right]\right)$

(a) $\frac{dF}{dt} = 100{,}000\left(\frac{2\pi}{365}\cos\left[\frac{2\pi(t - 60)}{365}\right]\right)$

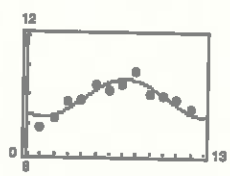
$= 0$ when $\frac{2\pi(t - 60)}{365} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

$t = 151.25$ or 333.75

The maximum occurs when $t = 151.25$ which corresponds to May 31.

(b) 

Sales are minimum when $t = 333.75$ which corresponds to November 30.

36. (a) 

(b) Maximum sales $x = 7.18$ (July)

(c) The cosine factor; 9.90

(d) $0.02t$ would mean a steady growth of consumption over time. In this case, the maximum consumption in 2000 (that is, on $73 \leq t \leq 84$) would be 12.1 billion gallons.

37. $R = 4.7t^4 - 193.5t^3 + 2941.7t^2 - 19{,}294.7t + 52{,}012$

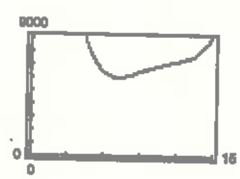
(a) $R'(t) = 18.8t^3 - 580.5t^2 + 5883.4t - 19{,}294.7 = 0$

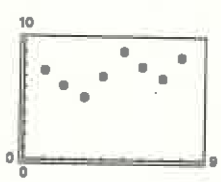
The real root is $t = 7.22$ (1987), which yields a minimum.

(b) The maximum occurs at $t = 14$ (1994).

(c) The minimum revenue was 5995 million dollars.

The maximum revenue was 8050.6 million dollars.

(d) 

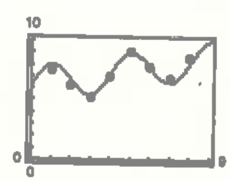
38. (a) 

(b) period $= 4 \Longrightarrow \beta = \pi/2$.

Using a graphing utility, $a + bt = 6.2 + 0.25t$.

Finally, $c \approx 1.5$

$$y = 6.2 + 0.25x + 1.5\sin\left(\frac{\pi}{2}x\right).$$

(c) 

(d) For the year 2000, $17 \leq t \leq 20$, and the maximum sales would be \$11.9 thousand.

39. (a) Demand function (b) Cost function

(c) Revenue function (d) Profit function

40. $P = -\frac{1}{10}s^3 + 6s^2 + 400$

(a) $\frac{dP}{ds} = -\frac{3}{10}s^2 + 12s = -\frac{3}{10}s(s - 40) = 0$ when $x = 0, s = 40$.

$\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12$

$\frac{d^2P}{ds^2}(0) > 0 \Longrightarrow s = 0$ yields a minimum.

$\frac{d^2P}{ds^2}(40) < 0 \Longrightarrow s = 40$ yields a maximum.

The maximum profit occurs when $s = 40$, which corresponds to \$40,000 ($P = \$3,600,000$).

(b) $\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12 = 0$ when $s = 20$.

The point of diminishing returns occurs when $s = 20$, which corresonds to \$20,000 being spent on advertising.

41. $\eta = \frac{p/x}{dp/dx} = \frac{(400 - 3x)/x}{-3} = 1 - \frac{400}{3x}$

When $x = 20$, we have

$$\eta = 1 - \frac{400}{3(20)} = -\frac{17}{3}.$$

Since $|\eta| = \frac{17}{3} > 1$, the demand is elastic.

42. $\eta = \frac{p/x}{dp/dx} = \frac{(5 - 0.03x)/x}{-0.03} = 1 - \frac{5}{0.03x}$

When $x = 100$, we have

$$\eta = 1 - \frac{5}{0.03(100)} = -\frac{2}{3}.$$

Since $|\eta| = \frac{2}{3} < 1$, the demand is inelastic.

43. $\eta = \frac{p/x}{dp/dx} = \frac{(400 - 0.5x^2)/x}{-x} = \frac{1}{2} - \frac{400}{x^2}$

When $x = 20$, we have

$$\eta = \frac{1}{2} - \frac{400}{(20)^2} = -\frac{1}{2}.$$

Since $|\eta| = \frac{1}{2} < 1$, the demand is inelastic.

44. $\eta = \frac{p/x}{dp/dx} = \frac{500/[x(x + 2)]}{-500/[(x + 2)^2]} = -\frac{x + 2}{x}$

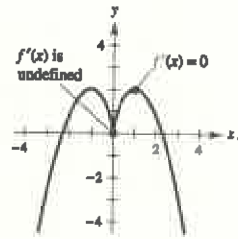
When $x = 23$, we have

$$\eta = -\frac{25}{23}.$$

Since $|\eta| = \frac{25}{23} > 1$, the demand is elastic.

Review Exercises for Chapter 3

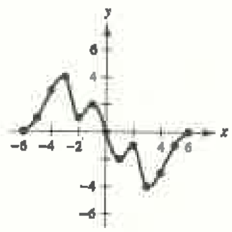
1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c.



2. (a) $f(4) = -f(-4) = -3$

(b) $f(-3) = -f(3) = -(-4) = 4$

(c)



At least six critical numbers on $(-6, 6)$.

(d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

(e) No, $\lim_{x \to 0} f(x)$ exists because f is continuous at $(0, 0)$.

(f) Yes, f is differentiable at $x = 2$.

3. $g(x) = 2x + 5\cos x, [0, 2\pi]$

$g'(x) = 2 - 5\sin x$

$= 0$ when $\sin x = \frac{2}{5}$.

Critical numbers: $x \approx 0.41, x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum

4. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}, [0, 2]$

$$f'(x) = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

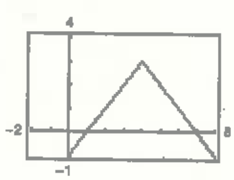
No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 2/\sqrt{5})$ Maximum

5. $f(x) = 3 - |x - 4|$

(a)



$f(1) = f(7) = 0$

(b) f is not differentiable at $x = 4$.

6. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

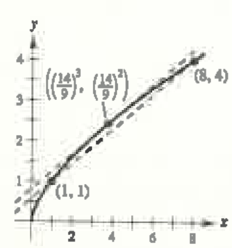
7. $f(x) = x^{2/3}, 1 \le x \le 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$



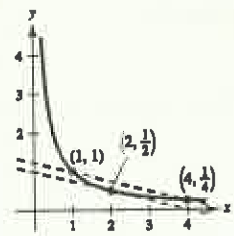
8. $f(x) = \frac{1}{x},\ 1 \le x \le 4$

$f'(x) = -\frac{1}{x^2}$

$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$

$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$

$c = 2$



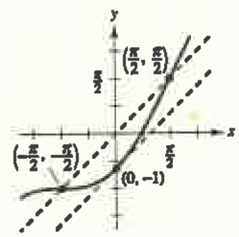
9. $f(x) = x - \cos x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$f'(x) = 1 + \sin x$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$f'(c) = 1 + \sin c = 1$

$c = 0$



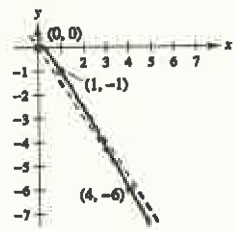
10. $f(x) = \sqrt{x} - 2x,\ 0 \le x \le 4$

$f'(x) = \frac{1}{2\sqrt{x}} - 2$

$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$

$c = 1$



11. $f(x) = Ax^2 + Bx + C$

$f'(x) = 2Ax + B$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$f'(c) = 2Ac + B = A(x_1 + x_2) + B$

$2Ac = A(x_1 + x_2)$

$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$

12. $f(x) = 2x^2 - 3x + 1$

$f'(x) = 4x - 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$

$f'(c) = 4c - 3 = 5$

$c = 2 = \text{Midpoint of } [0, 4]$

13. $f(x) = (x - 1)^2(x - 3)$

$f'(x) = (x - 1)^2(1) + (x - 3)(2)(x - 1)$

$= (x - 1)(3x - 7)$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

Interval	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

14. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critital number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

15. $h(x) = \sqrt{x}(x-3) = x^{3/2} - 3x^{1/2}$

Domain: $[0, \infty)$

$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$

$= \frac{3}{2}x^{-1/2}(x-1) = \frac{3(x-1)}{2\sqrt{x}}$

Critical numbers: $x = 0, x = 1$

Interval	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$	$h'(x) < 0$	$h'(x) > 0$
Conclusion	Decreasing	Increasing

16. $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

17. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$	$h'(t) < 0$	$h'(t) > 0$
Conclusion	Decreasing	Increasing

18. $g(x) = \frac{3}{2}\sin\left(\frac{\pi x}{2} - 1\right),\ [0, 4]$

$g'(x) = \frac{3}{2}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi x}{2} - 1\right)$

$= 0$ when $x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$

Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

19. $f(x) = x + \cos x,\ 0 \le x \le 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion	Concave downward	Concave upward	Concave downward

20. $f(x) = (x+2)^2(x-4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x = 0$ when $x = 0$.

Point of inflection: $(0, -16)$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

21. $\lim\limits_{x\to\infty} \dfrac{2x^2}{3x^2+5} = \lim\limits_{x\to\infty} \dfrac{2}{3+5/x^2} = \dfrac{2}{3}$

22. $\lim\limits_{x\to\infty} \dfrac{2x}{3x^2+5} = \lim\limits_{x\to\infty} \dfrac{2/x}{3+5/x^2} = 0$

23. $\lim\limits_{x\to\infty} \dfrac{5\cos x}{x} = 0$, since $|5\cos x| \le 5$.

24. $\lim\limits_{x\to\infty} \dfrac{3x}{\sqrt{x^2+4}} = \lim\limits_{x\to\infty} \dfrac{3}{\sqrt{1+4/x^2}} = 3$

25. $h(x) = \dfrac{2x+3}{x-4}$

Discontinuity: $x = 4$

$$\lim_{x\to\infty} \frac{2x+3}{x-4} = \lim_{x\to\infty} \frac{2+(3/x)}{1-(4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

26. $g(x) = \dfrac{5x^2}{x^2+2}$

$$\lim_{x\to\infty} \frac{5x^2}{x^2+2} = \lim_{x\to\infty} \frac{5}{1+(2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

27. $f(x) = \dfrac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x\to\infty} \left(\frac{3}{x} - 2\right) = -2$$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

28. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{x\to\infty} \frac{3x/x}{\sqrt{x^2+2}/\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{1+(2/x^2)}} = 3$$

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{x\to-\infty} \frac{3x/x}{\sqrt{x^2+2}/(-\sqrt{x^2})}$$

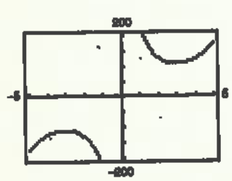
$$= \lim_{x\to-\infty} \frac{3}{-\sqrt{1+(2/x^2)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

29. $f(x) = x^3 + \dfrac{243}{x}$

Relative minimum: $(3, 108)$

Relative maximum: $(-3, -108)$

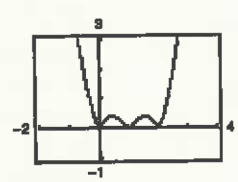


Vertical asymptote: $x = 0$

30. $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0)$, $(1, 0)$, $(2, 0)$

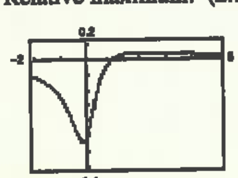
Relative maxima: $(1.577, 0.38)$, $(0.423, 0.38)$



31. $f(x) = \dfrac{x-1}{1+3x^2}$

Relative minimum: $(-0.155, -1.077)$

Relative maximum: $(2.155, 0.077)$

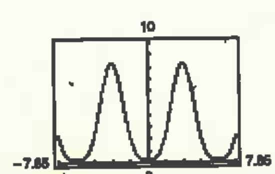


Horizontal asymptote: $y = 0$

32. $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

Relative minima: $(2\pi k, 0.29)$ where k is any integer.

Relative maxima: $((2k-1)\pi, 8.29)$ where k is any integer.



33. $f(x) = 4x - x^2 = x(4 - x)$

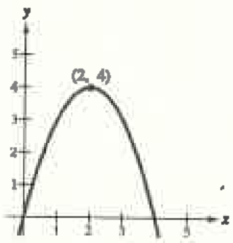
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4)$

$f'(x) = 4 - 2x = 0$ when $x = 2$.

$f''(x) = -2$

Therefore, $(2, 4)$ is a relative maximum.

Intercepts: $(0, 0)$, $(4, 0)$



34. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0$ when $x = 0, 3$.

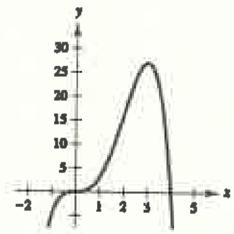
$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0$ when $x = 0, 2$.

$f''(3) < 0$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0)$, $(2, 16)$

Intercepts: $(0, 0)$, $(4, 0)$



35. $f(x) = x\sqrt{16 - x^2}$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$f'(x) = \dfrac{16 - 2x^2}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$ and undefined when $x = \pm 4$.

$f''(x) = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$

$f''(-2\sqrt{2}) > 0$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

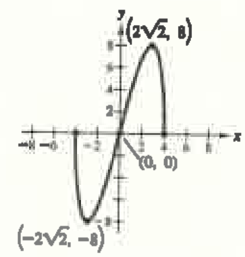
$f''(2\sqrt{2}) < 0$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin



36. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = 4x(x^2 - 4) = 0$ when $x = 0, \pm 2$.

$f''(x) = 4(3x^2 - 4) = 0$ when $x = \pm\dfrac{2\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 16)$ is a relative maximum.

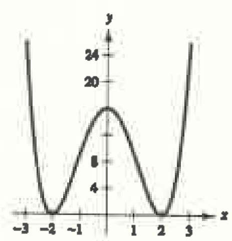
$f''(\pm 2) > 0$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0)$, $(0, 16)$, $(2, 0)$

Symmetry with respect to y-axis



37. $f(x) = (x-1)^3(x-3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = (x-1)^2(x-3)(5x-11) = 0$ when $x = 1, \dfrac{11}{5}, 3.$

$f''(x) = 4(x-1)(5x^2-22x+23) = 0$ when $x = 1, \dfrac{11 \pm \sqrt{6}}{5}.$

$f''(3) > 0$

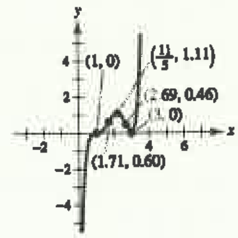
Therefore, $(3, 0)$ is a relative minimum.

$f''\left(\dfrac{11}{5}\right) < 0$

Therefore, $\left(\dfrac{11}{4}, \dfrac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0), \left(\dfrac{11-\sqrt{6}}{5}, 0.60\right), \left(\dfrac{11+\sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9), (1, 0), (3, 0)$



38. $f(x) = (x-3)(x+2)^3$

Domain: $(-\infty, \infty)$; Range: $\left[-\frac{16,875}{256}, \infty\right)$

$f'(x) = (x-3)(3)(x+2)^2 + (x+2)^3$

$\quad = (4x-7)(x+2)^2 = 0$ when $x = -2, \frac{7}{4}.$

$f''(x) = (4x-7)(2)(x+2) + (x+2)^2(4)$

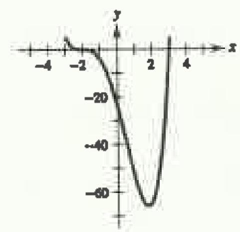
$\quad = 6(2x-1)(x+2) = 0$ when $x = -2, \frac{1}{2}.$

$f''\left(\frac{7}{4}\right) > 0$

Therefore, $\left(\frac{7}{4}, -\frac{16,875}{256}\right)$ is a relative minimum.

Points of inflection: $(-2, 0), \left(\frac{1}{2}, -\frac{625}{16}\right)$

Intercepts: $(-2, 0), (0, -24), (3, 0)$



39. $f(x) = x^{1/3}(x+3)^{2/3}$

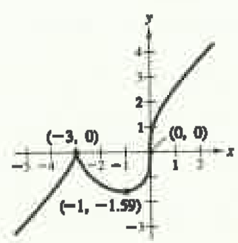
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = \dfrac{x+1}{(x+3)^{1/3}x^{2/3}} = 0$ when $x = -1$ and undefined when $x = -3, 0.$

$f''(x) = \dfrac{-2}{x^{5/3}(x+3)^{4/3}}$ is undefined when $x = 0, -3.$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $\left(-1, -\sqrt[3]{4}\right)$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0), (0, 0)$



40. $f(x) = (x-2)^{1/3}(x+1)^{2/3}$

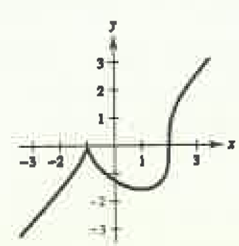
Graph of Exercise 39 translated 2 units to the right (x replaces by $x-2$).

$(-1, 0)$ is a relative maximum.

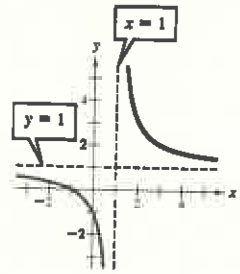
$\left(1, -\sqrt[3]{4}\right)$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$



41. $f(x) = \dfrac{x+1}{x-1}$



Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

$f'(x) = \dfrac{-2}{(x-1)^2} < 0$ if $x \neq 1$.

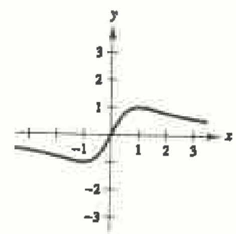
$f''(x) = \dfrac{4}{(x-1)^3}$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$

42. $f(x) = \dfrac{2x}{1+x^2}$



Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$f'(x) = \dfrac{2(1-x)(1+x)}{(1+x^2)^2} = 0$ when $x = \pm 1$.

$f''(x) = \dfrac{-2x(3-x^2)}{(1+x^2)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

$f''(1) < 0$

Therefore, $(1, 1)$ is a relative maximum.

$f''(-1) > 0$

Therefore, $(-1, -1)$ is a relative minimum.

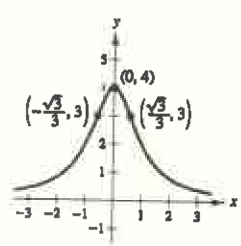
Points of inflection: $\left(-\sqrt{3}, -\sqrt{3}/2\right), (0, 0), \left(\sqrt{3}, \sqrt{3}/2\right)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$

43. $f(x) = \dfrac{4}{1+x^2}$



Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$f'(x) = \dfrac{-8x}{(1+x^2)^2} = 0$ when $x = 0$.

$f''(x) = \dfrac{-8(1-3x^2)}{(1+x^2)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 4)$ is a relative maximum.

Points of inflection: $\left(\pm\sqrt{3}/3, 3\right)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$

44. $f(x) = \dfrac{x^2}{1 + x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \dfrac{1}{2}\right]$

$f'(x) = \dfrac{(1 + x^4)(2x) - x^2(4x^3)}{(1 + x^4)^2} = \dfrac{2x(1 - x)(1 + x)(1 + x^2)}{(1 + x^4)^2} = 0$ when $x = 0, \pm 1$.

$f''(x) = \dfrac{(1 + x^4)^2(2 - 10x^4) - (2x - 2x^5)(2)(1 + x^4)(4x^3)}{(1 + x^4)^4} = \dfrac{2(1 - 12x^4 + 3x^8)}{(1 + x^4)^3} = 0$ when $x = \pm \sqrt[4]{\dfrac{6 \pm \sqrt{33}}{3}}$

$f''(\pm 1) < 0$

Therefore, $\left(\pm 1, \dfrac{1}{2}\right)$ are relative maxima.

$f''(0) > 0$

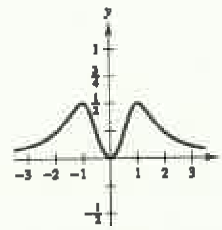
Therefore, $(0, 0)$ is a relative minimum.

Points of inflection: $\left(\pm \sqrt[4]{\dfrac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm \sqrt[4]{\dfrac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



45. $f(x) = x^3 + x + \dfrac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$f'(x) = 3x^2 + 1 - \dfrac{4}{x^2} = \dfrac{3x^4 + x^2 - 4}{x^2} = 0$ when $x = \pm 1$.

$f''(x) = 6x + \dfrac{8}{x^3} = \dfrac{6x^4 + 8}{x^3} \neq 0$

$f''(-1) < 0$

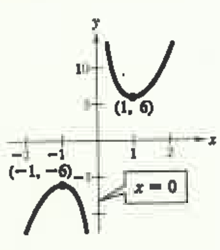
Therefore, $(-1, -6)$ is a relative maximum.

$f''(1) > 0$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin



46. $f(x) = x^2 + \dfrac{1}{x} = \dfrac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = 2x - \dfrac{1}{x^2} = \dfrac{2x^3 - 1}{x^2} = 0$ when $x = \dfrac{1}{\sqrt[3]{2}}$.

$f''(x) = 2 + \dfrac{2}{x^3} = \dfrac{2(x^3 + 1)}{x^3} = 0$ when $x = -1$.

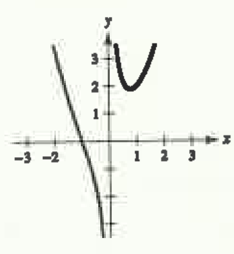
$f''\left(\dfrac{1}{\sqrt[3]{2}}\right) > 0$

Therefore, $\left(\dfrac{1}{\sqrt[3]{2}}, \dfrac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$



47. $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = \dfrac{2x(x^2 - 9)}{|x^2 - 9|} = 0$ when $x = 0$ and is undefined when $x = \pm 3$.

$f''(x) = \dfrac{2(x^2 - 9)}{|x^2 - 9|}$ is undefined at $x = \pm 3$.

$f''(0) < 0$

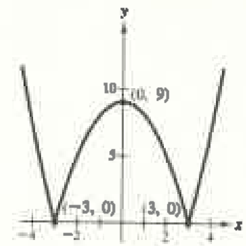
Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Points of inflection: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis

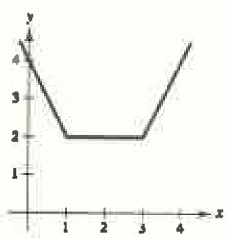


48. $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \le 1 \\ 2, & 1 < x \le 3 \\ 2x - 4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$



49. $f(x) = x + \cos x$

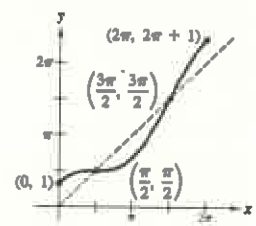
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$f'(x) = 1 - \sin x \ge 0$, f is increasing.

$f''(x) = -\cos x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, $\left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

Intercept: $(0, 1)$



50. $f(x) = \dfrac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[\dfrac{-3\sqrt{3}}{2\pi}, \dfrac{3\sqrt{3}}{2\pi}\right]$

$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$

Critical Numbers: $x = \pm\dfrac{2}{3}, 0$

$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0$ when $x = 0, \pm 1, \pm 0.420$.

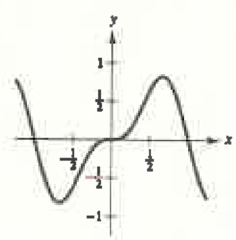
By the First Derivative Test: $\left(-\dfrac{2}{3}, \dfrac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\dfrac{2}{3}, \dfrac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

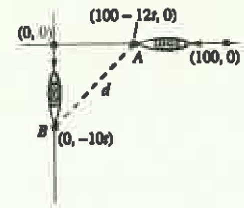
Points of inflection: $(-0.420, -0.462)$, $(0.420, 0.462)$, $(\pm 1, 0)$, $(0, 0)$

Intercepts: $(-1, 0)$, $(0, 0)$, $(1, 0)$

Symmetric with respect to the origin



51. Let $t = 0$ at noon.



$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10{,}000 - 2400t + 244t^2$$

$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

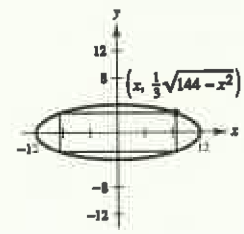
Ship A at (40.98, 0); Ship B at (0, −49.18)

$$d^2 = 10{,}000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4{:}55 \text{ P.M.}.$$

$$d \approx 64 \text{ km}$$

52. Ellipse: $\dfrac{x^2}{144} + \dfrac{y^2}{16} = 1$, $y = \dfrac{1}{3}\sqrt{144 - x^2}$



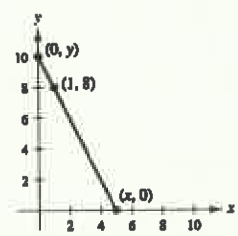
$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\frac{dA}{dx} = \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right]$$

$$= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.$$

The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \dfrac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

53. We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,



$$m = \frac{y - 8}{0 - 1} = \frac{0 - 8}{x - 1} \text{ or } y = \frac{8x}{x - 1}.$$

Let $f(x) = L^2 = x^2 + \left(\dfrac{8x}{x - 1}\right)^2$.

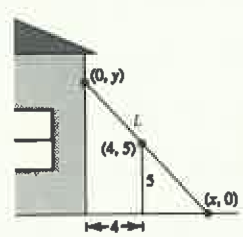
$$f'(x) = 2x + 128\left(\frac{x}{x - 1}\right)\left[\frac{(x - 1) - x}{(x - 1)^2}\right] = 0$$

$$x - \frac{64x}{(x - 1)^3} = 0$$

$$x[(x - 1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum).}$$

Vertices of triangle: (0, 0), (5, 0), (0, 10)

54. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,



$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

Let $f(x) = L^2 = x^2 + \left(\dfrac{5x}{x - 4}\right)^2$

$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4}\sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}}\sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$

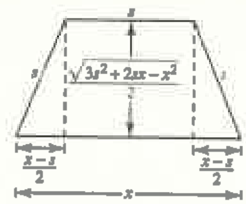
55. $A = (\text{Average of bases})(\text{Height})$

$$= \left(\frac{x+s}{2}\right)\frac{\sqrt{3s^2+2sx-x^2}}{2} \quad \text{(see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4}\left[\frac{(s-x)(s+x)}{\sqrt{3s^2+2sx-x^2}} + \sqrt{3s^2+2sx-x^2}\right]$$

$$= \frac{2(2s-x)(s+x)}{4\sqrt{3s^2+2sx-x^2}} = 0 \text{ when } x = 2s.$$

A is a maximum when $x = 2s$.

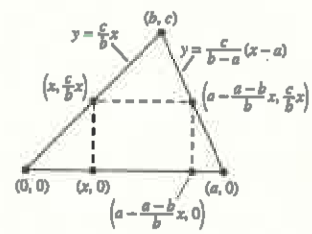


56. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c). The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b-a)](x-a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b-a}(\bar{x}-a)$$

$$(b-a)x = b(\bar{x}-a)$$

$$\bar{x} = \frac{b-a}{b}x + a = a - \frac{a-b}{b}x.$$



Finally, the lower right vertex is

$$\left(a - \frac{a-b}{b}x, 0\right).$$

Width of rectangle: $a - \dfrac{a-b}{b}x - x$

Height of rectangle: $\dfrac{c}{b}x$ (see figure)

$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a-b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

57. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$. Assume that the hypotenuse of length L passes through $(4, 6)$.

$$m = \frac{y-6}{0-4} = \frac{6-0}{4-x} \text{ or } y = \frac{6x}{x-4}$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\dfrac{6x}{x-4}\right)^2.$

$$f'(x) = 2x + 72\left(\frac{x}{x-4}\right)\left[\frac{-4}{(x-4)^2}\right] = 0$$

$$x[(x-4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

58. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y-b}{0-a} = \frac{b-0}{a-x} \Longrightarrow y = \frac{-bx}{a-x}.$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\dfrac{-bx}{a-x}\right)^2.$

$$f'(x) = 2x + 2\left(\frac{-bx}{a-x}\right)\left[\frac{-ab}{(a-x)^2}\right]$$

$$\frac{2x[(a-x)^3 + ab^2]}{(a-x)^3} = 0 \text{ when } x = 0,\ a + \sqrt[3]{ab^2}.$$

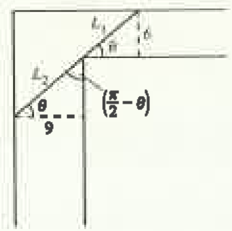
Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$L = \sqrt{\left(a + \sqrt[3]{ab^2}\right)^2 + \left(b + \sqrt[3]{a^2b}\right)^2}$$

$$= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2}$$

$$= (a^{2/3} + b^{2/3})^{3/2} \text{ meters}$$

59. $\csc\theta = \dfrac{L_1}{6}$ or $L_1 = 6\csc\theta$ (see figure)



$$\csc\left(\frac{\pi}{2}-\theta\right) = \frac{L_2}{9} \text{ or } L_2 = 9\csc\left(\frac{\pi}{2}-\theta\right)$$

$$L = L_1 + L_2 = 6\csc\theta + 9\csc\left(\frac{\pi}{2}-\theta\right) = 6\csc\theta + 9\sec\theta$$

$$\frac{dL}{d\theta} = -6\csc\theta\cot\theta + 9\sec\theta\tan\theta = 0$$

$$\tan^3\theta = \frac{2}{3} \Longrightarrow \tan\theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{3^{1/3}}$$

$$\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{2^{1/3}}$$

$$L = 6\frac{(3^{2/3}+2^{2/3})^{1/2}}{2^{1/3}} + 9\frac{(3^{2/3}+2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3}+2^{2/3})^{3/2}\text{ ft} \approx 21.07\text{ ft}$$ (Compare to Exercise 58 using $a = 9$ and $b = 6$.)

60. Using Exercise 59 as a guide we have $L_1 = a\csc\theta$ and $L_2 = b\sec\theta$. Then $dL/d\theta = -a\csc\theta\cot\theta + b\sec\theta\tan\theta = 0$ when

$$\tan\theta = \sqrt[3]{a/b},\ \sec\theta = \frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}},\ \csc\theta = \frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a\csc\theta + b\sec\theta = a\frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}} + b\frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3}+b^{2/3})^{3/2}.$$

This matches the result of Exercise 58.

61. $y = \dfrac{1}{3}\cos(12t) - \dfrac{1}{4}\sin(12t)$

$v = y' = -4\sin(12t) - 3\cos(12t)$

(a) When $t = \dfrac{\pi}{8}$, $y = \dfrac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4\sin(12t) - 3\cos(12t) = 0$ when $\dfrac{\sin(12t)}{\cos(12t)} = -\dfrac{3}{4} \Longrightarrow \tan(12t) = -\dfrac{3}{4}$.

Therefore, $\sin(12t) = -\dfrac{3}{5}$ and $\cos(12t) = \dfrac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12}\text{ inch.}$$

(c) Period: $\dfrac{2\pi}{12} = \dfrac{\pi}{6}$

Frequency: $\dfrac{1}{\pi/6} = \dfrac{6}{\pi}$

62. (a) $y = A\sin(\sqrt{k/m}\,t) + B\cos(\sqrt{k/m}\,t)$

$$y' = A\sqrt{k/m}\cos(\sqrt{k/m}\,t) - B\sqrt{k/m}\sin(\sqrt{k/m}\,t)$$

$$= 0 \text{ when } \frac{\sin\sqrt{k/m}\,t}{\cos\sqrt{k/m}\,t} = \frac{A}{B} \Rightarrow \tan(\sqrt{k/m}\,t) = \frac{A}{B}.$$

Therefore,

$$\sin(\sqrt{k/m}\,t) = \frac{A}{\sqrt{A^2+B^2}}$$

$$\cos(\sqrt{k/m}\,t) = \frac{B}{\sqrt{A^2+B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2+B^2}}\right) + B\left(\frac{B}{\sqrt{A^2+B^2}}\right) = \sqrt{A^2+B^2}.$$

(b) Period: $\dfrac{2\pi}{\sqrt{k/m}}$

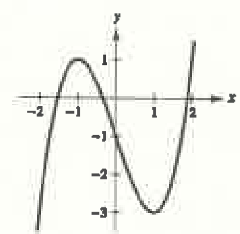
Frequency: $\dfrac{1}{2\pi/\sqrt{k/m}} = \dfrac{1}{2\pi}\sqrt{k/m}$

(c) The frequency is changed as the square root of k.

(d) The frequency is inversely proportional to the square root of the mass.

63. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.



$f'(x) = 3x^2 - 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

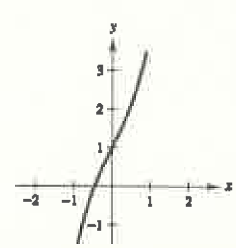
n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

64. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.



$f'(x) = 3x^2 + 2$

f changes sign in $[-1, 0]$.

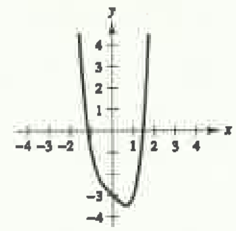
n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.1250	2.7500	-0.0455	-0.4545
2	-0.4545	-0.0029	2.6197	-0.0011	-0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.

65. Find the zeros of $f(x) = x^4 - x - 3$.

$f'(x) = 4x^3 - 1$

From the graph you can see that $f(x)$ has two real zeros.



f changes sign in $[-2, -1]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.2000	0.2736	−7.9120	−0.0346	−1.1654
2	−1.1654	0.0100	−7.3312	−0.0014	−1.1640

On the interval $[-2, -1]$: $x \approx -1.164$.

f changes sign in $[1, 2]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5000	0.5625	12.5000	0.0450	1.4550
2	1.4550	0.0268	11.3211	0.0024	1.4526
3	1.4526	−0.0003	11.2602	0.0000	1.4526

On the interval $[1, 2]$; $x \approx 1.453$.

66. Find the zeros of $f(x) = \sin \pi x + x - 1$.

$f'(x) = \pi \cos \pi x + 1$

From the graph you can see that $f(x)$ has three real zeros.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.2000	−0.2122	3.5416	−0.0599	0.2599
2	0.2599	−0.0113	3.1513	−0.0036	0.2635
3	0.2635	0.0000	3.1253	0.0000	0.2635

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	0.0000	−2.1416	0.0000	1.0000

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.8000	0.2122	3.5416	0.0599	1.7401
2	1.7401	0.0113	3.1513	0.0036	1.7365
3	1.7365	0.0000	3.1253	0.0000	1.7365

The three real zeros of $f(x)$ are $x \approx 0.264$, $x = 1$, and $x \approx 1.737$.

67. $y = x(1 - \cos x) = x - x\cos x$

$$\frac{dy}{dx} = 1 + x\sin x - \cos x$$

$$dy = (1 + x\sin x - \cos x)\,dx$$

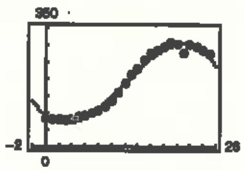
68. $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}}\,dx$$

69. (a) $S = -0.0810t^3 + 2.9197t^2 - 15.5459t + 92.3247$

(b)

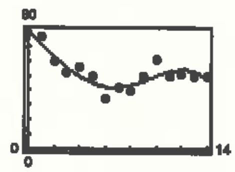


(c) Maximum is $S = 303.3$ at $t = 20.98$ (1990). Note that this is not accurate!

(d) The derivative $S'(t)$ is greatest at $t = 12.0$ (1982).

70. (a) $I = -0.009696t^4 + 0.2123t^3 - 0.7089t^2 - 7.3904t + 79.1361$

(b)



(c) According to the model, the cost was a minimum (42.1) at $t = 6.7$ (1986).

(d) $I'(t)$ is decreasing most rapidly when $t = 14$ (1994).

71.

$$S = 4\pi r^2\; dr = \Delta r = \pm 0.025$$

$$dS = 8\pi r\,dr = 8\pi(9)(\pm 0.025)$$

$$= \pm 1.8\pi \text{ square cm}$$

$$\frac{dS}{S}(100) = \frac{8\pi r\,dr}{4\pi r^2}(100) = \frac{2\,dr}{r}(100)$$

$$= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\%$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\,dr = 4\pi(9)^2(\pm 0.025)$$

$$= \pm 8.1\pi \text{ cubic cm}$$

$$\frac{dV}{V}(100) = \frac{4\pi r^2\,dr}{(4/3)\pi r^3}(100) = \frac{3\,dr}{r}(100)$$

$$= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\%$$

72. $p = 75 - \frac{1}{4}x$

$$\Delta p = p(8) - p(7)$$

$$= \left(75 - \frac{8}{4}\right) - \left(75 - \frac{7}{4}\right) = -\frac{1}{4}$$

$$dp = -\frac{1}{4}dx = -\frac{1}{4}(1) = -\frac{1}{4}$$

73. $p = 36 - 4x$

$$C = 2x^2 + 6$$

$$R = xp$$

$$P = R - C = x(36 - 4x) - (2x^2 + 6) = -6(x^2 - 6x + 1)$$

$$\frac{dP}{dx} = -6(2x - 6) = 0 \text{ when } x = 3.$$

$$\frac{d^2P}{dx^2} = -12 < 0,\; x = 3 \text{ is a maximum.}$$

For $x = 3$, the profit is $P = \$48$.

74. $C = \frac{1}{4}x^2 + 62x + 125$

$p = 75 - \frac{1}{3}x$

$R = xp$

(a) $P = R - C = x\left(75 - \frac{1}{3}x\right) - \left(\frac{1}{4}x^2 + 62x + 125\right)$

$\frac{dP}{dx} = -\frac{1}{3}x + 75 - \frac{1}{3}x - \frac{1}{2}x - 62 = -\frac{7}{6}x + 13 = 0$ when $x = \frac{78}{7}$ (11 units).

(b) $\overline{C}(x) = \frac{C}{x} = \frac{1}{4}x + 62 + \frac{125}{x}$ (average cost)

$\frac{d\overline{C}}{dx} = \frac{1}{4} - \frac{125}{x^2} = 0$

$x^2 - 500 = 0$ when $x = 10\sqrt{5}$ (22 units).

(c) $\eta = \frac{p/x}{dp/dx} = \frac{(75/x) - (1/3)}{-1/3} = 1 - \frac{225}{x} = \frac{x - 255}{x}$

75. R = (Number of people)(Rate per person)

$= n[8.00 - 0.05(n - 80)],\ n \geq 80$

$= 12n - 0.05n^2$

$\frac{dR}{dn} = 12 - 0.10n = 0$ when $n = 120$.

$\frac{d^2R}{dn^2} = -0.10 < 0 \Rightarrow$ Revenue will be maximum when 120 people go on the bus.

76. The cost of fuel is proportional to $s^{3/2}$, $C_F = 50$ when $s = 25$, and fixed cost are \$100.

$C_F = ks^{3/2}$ where $50 = k(25)^{3/2} \Rightarrow k = \frac{2}{5}$.

The total cost $C = \frac{2}{5}s^{3/2} + 100$.

$\overline{C} = \frac{C}{s} = \frac{(2/5)s^{3/2} + 100}{s} = \frac{2}{5}s^{1/2} + \frac{100}{s}$

$\frac{d\overline{C}}{ds} = \frac{1}{5s^{1/2}} - \frac{100}{s^2} = 0$ when $s^{3/2} = 500$.

$s = 500^{2/3} \approx 63$ mi/hr

77. $C = \left(\frac{Q}{x}\right)s + \left(\frac{x}{2}\right)r$

$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$

$\frac{Qs}{x^2} = \frac{r}{2}$

$x^2 = \frac{2Qs}{r}$

$x = \sqrt{\frac{2Qs}{r}}$

78. $p = 600 - 3x$

$C = 0.3x^2 + 6x + 600$

$P = xp - C - xt$

$= x(600 - 3x) - (0.3x^2 + 600) - xt$

$= -3.3x^2 + (594 - t)x - 600$

$\frac{dP}{dx} = -6.6x + 594 - t = 0$

$x = \frac{594 - t}{6.6}$

(a) $t = 5$, $x = 89$, $P = 25{,}681.70$

(b) $t = 10$, $x = 88$, $P = 25{,}236.80$

(c) $t = 20$, $x = 87$, $P = 24{,}360.30$

79. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

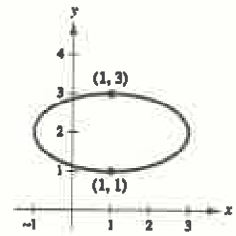
$$(x-1)^2 + 4(y-2)^2 = 4$$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$



(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$2x + 8y\frac{dy}{dx} - 2 - 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2-2x}{8y-16} = \frac{1-x}{4y-8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

80. $f(x) = x^n$, n is a positive integer.

(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

(b) $f''(x) = n(n-1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when n is odd and $n \geq 3$.

81. False

Let $f(x) = x^3$, $c = 0$.

82. False

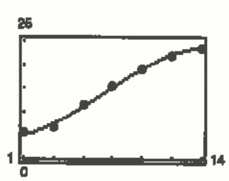
Let $f(x) = \dfrac{2x}{\sqrt{x^2+2}}$.

f has horizontal asymptotes at $y = 2$ and $y = -2$.

83. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

84. (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$

(b)



(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the t^3 coefficient term is negative.

CHAPTER 4
Integration

CHAPTER 4
Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Exercises

1. $\frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$

2. $\frac{d}{dx}\left(x^4 + \frac{1}{x} + C\right) = 4x^3 - \frac{1}{x^2}$

3. $\frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x - 2)(x + 2)$

4. $\frac{d}{dx}\left(\frac{2(x^2 + 3)}{3\sqrt{x}} + C\right) = \frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{-1/2} + C\right)$

$= x^{1/2} - x^{-3/2} = \frac{x^2 - 1}{x^{3/2}}$

	Given	Rewrite	Integrate	Simplify
5.	$\int \sqrt[3]{x}\,dx$	$\int x^{1/3}\,dx$	$\frac{x^{4/3}}{4/3} + C$	$\frac{3}{4}x^{4/3} + C$
6.	$\int \frac{1}{x^2}\,dx$	$\int x^{-2}\,dx$	$\frac{x^{-1}}{-1} + C$	$-\frac{1}{x} + C$
7.	$\int \frac{1}{x\sqrt{x}}\,dx$	$\int x^{-3/2}\,dx$	$\frac{x^{-1/2}}{-1/2} + C$	$-\frac{2}{\sqrt{x}} + C$
8.	$\int x(x^2 + 3)\,dx$	$\int (x^3 + 3x)\,dx$	$\frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C$	$\frac{1}{4}x^4 + \frac{3}{2}x^2 + C$
9.	$\int \frac{1}{2x^3}\,dx$	$\frac{1}{2}\int x^{-3}\,dx$	$\frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + C$	$-\frac{1}{4x^2} + C$
10.	$\int \frac{1}{(2x)^3}\,dx$	$\frac{1}{8}\int x^{-3}\,dx$	$\frac{1}{8}\left(\frac{x^{-2}}{-2}\right) + C$	$-\frac{1}{16x^2} + C$

11. $\frac{dy}{dt} = 3t^2$

$y = t^3 + C$

Check: $\frac{d}{dt}[t^3 + C] = 3t^2$

12. $\frac{dr}{d\theta} = \pi$

$r = \pi\theta + C$

Check: $\frac{d}{d\theta}[\pi\theta + C] = \pi$

13. $\frac{dy}{dx} = x^{3/2}$

$y = \frac{2}{5}x^{5/2} + C$

Check: $\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$

14. $\frac{dy}{dx} = 3x^{-4}$

$y = -x^{-3} + C = -\frac{1}{x^3} + C$

Check: $\frac{d}{dx}[-x^{-3} + C] = 3x^{-4}$

15. $\int (x^3 + 2)\,dx = \frac{1}{4}x^4 + 2x + C$

Check: $\frac{d}{dx}\left(\frac{1}{4}x^4 + 2x + C\right) = x^3 + 2$

16. $\int (x^2 - 2x + 3)\,dx = \frac{1}{3}x^3 - x^2 + 3x + C$

Check: $\frac{d}{dx}\left(\frac{1}{3}x^3 - x^2 + 3x + C\right) = x^2 - 2x + 3$

17. $\int (x^{3/2} + 2x + 1)\,dx = \frac{2}{5}x^{5/2} + x^2 + x + C$

Check: $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$

18. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right) dx = \frac{x^{3/2}}{3/2} + \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \frac{2}{3}x^{3/2} + x^{1/2} + C$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \frac{1}{2}x^{-1/2} = \sqrt{x} + \frac{1}{2\sqrt{x}}$

19. $\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$

Check: $\frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$

20. $\int (\sqrt[4]{x^3} + 1)\,dx = \int (x^{3/4} + 1)\,dx = \frac{4}{7}x^{7/4} + x + C$

Check: $\frac{d}{dx}\left(\frac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$

21. $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$

Check: $\frac{d}{dx}\left(-\frac{1}{2x^2} + C\right) = \frac{1}{x^3}$

22. $\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$

Check: $\frac{d}{dx}\left(-\frac{1}{3x^3} + C\right) = \frac{1}{x^4}$

23. $\int \frac{x^2 + x + 1}{\sqrt{x}}\,dx = \int (x^{3/2} + x^{1/2} + x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$

Check: $\frac{d}{dx}\left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C\right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$

24. $\int \frac{x^2 + 1}{x^2}\,dx = \int (1 + x^{-2})\,dx$

$= x + \frac{x^{-1}}{-1} + C = x - \frac{1}{x} + C$

Check: $\frac{d}{dx}\left(x - \frac{1}{x} + C\right) = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$

25. $\int (x + 1)(3x - 2)\,dx = \int (3x^2 + x - 2)\,dx$

$= x^3 + \frac{1}{2}x^2 - 2x + C$

Check: $\frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - 2x + C\right) = 3x^2 + x - 2$

$= (x + 1)(3x - 2)$

26. $\int (2t^2 - 1)^2\,dt = \int (4t^4 - 4t^2 + 1)\,dt$

$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$

Check: $\frac{d}{dt}\left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C\right) = 4t^4 - 4t^2 + 1$

$= (2t^2 - 1)^2$

27. $\int y^2\sqrt{y}\,dy = \int y^{5/2}\,dy = \frac{2}{7}y^{7/2} + C$

Check: $\frac{d}{dy}\left(\frac{2}{7}y^{7/2} + C\right) = y^{5/2} = y^2\sqrt{y}$

28. $\int (1 + 3t)t^2\,dt = \int (t^2 + 3t^3)\,dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C\right) = t^2 + 3t^3 = (1 + 3t)t^2$

29. $\int dx = \int 1\,dx = x + C$

Check: $\frac{d}{dx}(x + C) = 1$

30. $\displaystyle\int 3\,dt = 3t + C$

Check: $\dfrac{d}{dt}(3t + C) = 3$

31. $\displaystyle\int (2\sin x + 3\cos x)\,dx = -2\cos x + 3\sin x + C$

Check: $\dfrac{d}{dx}(-2\cos x + 3\sin x + C) = 2\sin x + 3\cos x$

32. $\displaystyle\int (t^2 - \sin t)\,dt = \frac{1}{3}t^3 + \cos t + C$

Check: $\dfrac{d}{dt}\left(\dfrac{1}{3}t^3 + \cos t + C\right) = t^2 - \sin t$

33. $\displaystyle\int (1 - \csc t\cot t)\,dt = t + \csc t + C$

Check: $\dfrac{d}{dt}(t + \csc t + C) = 1 - \csc t\cot t$

34. $\displaystyle\int (\theta^2 + \sec^2\theta)\,d\theta = \frac{1}{3}\theta^3 + \tan\theta + C$

Check: $\dfrac{d}{d\theta}\left(\dfrac{1}{3}\theta^3 + \tan\theta + C\right) = \theta^2 + \sec^2\theta$

35. $\displaystyle\int (\sec^2\theta - \sin\theta)\,d\theta = \tan\theta + \cos\theta + C$

Check: $\dfrac{d}{d\theta}(\tan\theta + \cos\theta + C) = \sec^2\theta - \sin\theta$

36. $\displaystyle\int \sec y(\tan y - \sec y)\,dy = \int (\sec y\tan y - \sec^2 y)\,dy$

$= \sec y - \tan y + C$

Check: $\dfrac{d}{dy}(\sec y - \tan y + C) = \sec y\tan y - \sec^2 y$

$= \sec y(\tan y - \sec y)$

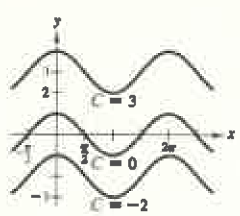
37. $\displaystyle\int (\tan^2 y + 1)\,dy = \int \sec^2 y\,dy = \tan y + C$

Check: $\dfrac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$

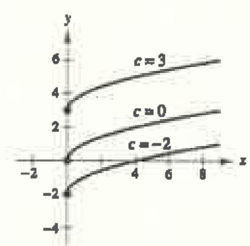
38. $\displaystyle\int \frac{\sin x}{1 - \sin^2 x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos}\right)\left(\frac{\sin x}{\cos x}\right)dx = \int \sec x\tan x\,dx = \sec x + C$

Check: $\dfrac{d}{dx}(\sec x + C) = \sec x\tan x = \dfrac{\sin x}{\cos^2 x} = \dfrac{\sin x}{1 - \sin^2 x}$

39. $f(x) = \cos x$

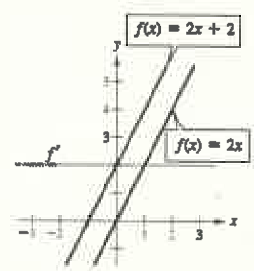


40. $f(x) = \sqrt{x}$



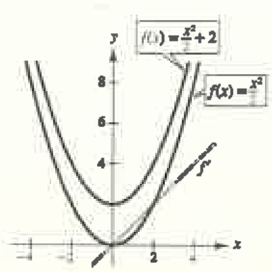
41. $f'(x) = 2$

$f(x) = 2x + C$



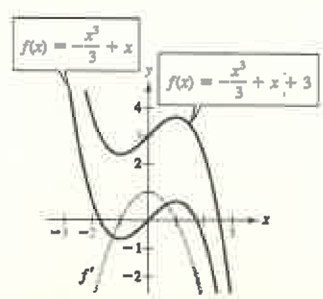
42. $f'(x) = x$

$f(x) = \dfrac{x^2}{2} + C$



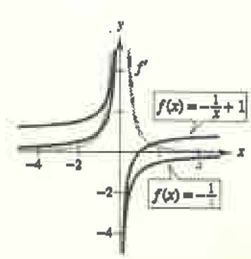
43. $f'(x) = 1 - x^2$

$f(x) = x - \dfrac{x^3}{3} + C$



44. $f'(x) = \dfrac{1}{x^2}$

$f(x) = -\dfrac{1}{x} + C$



45. $\dfrac{dy}{dx} = 2x - 1, \ (1, 1)$

$$y = \int (2x - 1)\,dx = x^2 - x + C$$

$$1 = (1)^2 - (1) + C \Rightarrow C = 1$$

$$y = x^2 - x + 1$$

46. $\dfrac{dy}{dx} = 2(x - 1) = 2x - 2, \ (3, 2)$

$$y = \int 2(x - 1)\,dx = x^2 - 2x + C$$

$$2 = (3)^2 - 2(3) + C \Rightarrow C = -1$$

$$y = x^2 - 2x - 1$$

47. $\dfrac{dy}{dx} = \cos x, \ (0, 4)$

$$y = \int \cos x\,dx = \sin x + C$$

$$4 = \sin 0 + C \Rightarrow C = 4$$

$$y = \sin x + 4$$

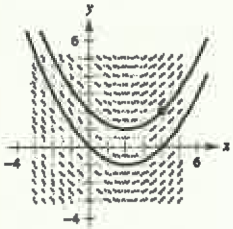
48. $\dfrac{dy}{dx} = -\dfrac{1}{x^2} = -x^{-2}, \ (1,3)$

$$y = \int -x^{-2}\,dx = \frac{1}{x} + C$$

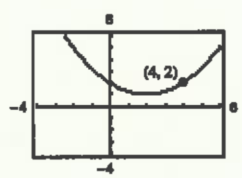
$$3 = \frac{1}{1} + C \Rightarrow C = 2$$

$$y = \frac{1}{x} + 2, \ x > 0$$

49. (a)



(b) $\dfrac{dy}{dx} = \dfrac{1}{2}x - 1, \ (4, 2)$



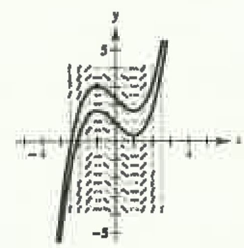
$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

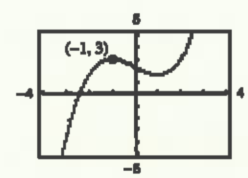
$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$

50. (a)



(b) $\dfrac{dy}{dx} = x^2 - 1, \ (-1, 3)$



$$y = \frac{x^3}{3} - x + C$$

$$3 = \frac{(-1)^3}{3} - (-1) + C$$

$$3 = -\frac{1}{3} + 1 + C$$

$$C = \frac{7}{3}$$

$$y = \frac{x^3}{3} - x + \frac{7}{3}$$

51. $f''(x) = 2$

$f'(2) = 5$

$f(2) = 10$

$$f'(x) = \int 2\,dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1)\,dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$$

$$f(x) = x^2 + x + 4$$

52. $f''(x) = x^2$

$f'(0) = 6$

$f(0) = 3$

$$f'(x) = \int x^2\,dx = \frac{1}{3}x^3 + C_1$$

$$f'(0) = 0 + C_1 = 6 \Rightarrow C_1 = 6$$

$$f'(x) = \frac{1}{3}x^3 + 6$$

$$f(x) = \int \left(\frac{1}{3}x^3 + 6\right)dx = \frac{1}{12}x^4 + 6x + C_2$$

$$f(0) = 0 + 0 + C_2 = 3 \Rightarrow C_2 = 3$$

$$f(x) = \frac{1}{12}x^4 + 6x + 3$$

53. $f''(x) = x^{-3/2}$

$f'(4) = 2$

$f(0) = 0$

$f'(x) = \int x^{-3/2}\,dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$

$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$

$f'(x) = -\frac{2}{\sqrt{x}} + 3$

$f(x) = \int (-2x^{-1/2} + 3)\,dx = -4x^{1/2} + 3x + C_2$

$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$

$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$

54. $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$f'(x) = \int \sin x\,dx = -\cos x + C_1$

$f'(0) = -1 + C_1 = 1 \Rightarrow C_1 = 2$

$f'(x) = -\cos x + 2$

$f(x) = \int (-\cos x + 2)\,dx = -\sin x + 2x + C_2$

$f(0) = 0 + 0 + C_2 = 6 \Rightarrow C_2 = 6$

$f(x) = -\sin x + 2x + 6$

55. (a) $h(t) = \int (1.5t + 5)\,dt = 0.75t^2 + 5t + C$

$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$

$h(t) = 0.75t^2 + 5t + 12$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69$ cm

56. $\frac{dP}{dt} = k\sqrt{t},\ 0 \le t \le 10$

$P(t) = \int kt^{1/2}\,dt = \frac{2}{3}kt^{3/2} + C$

$P(0) = 0 + C = 500 \Rightarrow C = 500$

$P(1) = \frac{2}{3}k + 500 = 600 \Rightarrow k = 150$

$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$

$P(7) = 100(7)^{3/2} + 500 \approx 2352$ bacteria

57. $a(t) = -32$ ft/sec^2

$v(t) = \int -32\,dt = -32t + C_1$

$v(0) = 60 = C_1$

$v(t) = -32t + 60$

$s(t) = \int (-32t + 60)\,dt = -16t^2 + 60t + C_2$

$s(0) = 0 = C_2$

$s(t) = -16t^2 + 60t$

The ball reaches its maximum height when

$v(t) = 0$: $-32t + 60 = 0$

$32t = 60$

$t = \frac{15}{8}$.

$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) = 56.26$ feet

58. $f''(t) = a(t) = -32$ ft/sec^2

$f'(0) = v_0$

$f(0) = s_0$

$f'(t) = v(t) = \int -32\,dt = -32t + C_1$

$f'(0) = 0 + C_1 = v_0 \Rightarrow C_1 = v_0$

$f'(t) = -32t + v_0$

$f(t) = s(t) = \int (-32t + v_0)\,dt = -16t^2 + v_0t + C_2$

$f(0) = 0 + 0 + C_2 = s_0 \Rightarrow C_2 = s_0$

$f(t) = -16t^2 + v_0t + s_0$

59. From Exercise 58, we have:

$$s(t) = -16t^2 + v_0t$$

$$s'(t) = -32t + v_0 = 0 \text{ when } t = \frac{v_0}{32} = \text{time to reach maximum height.}$$

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35{,}200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

60. $v_0 = 16$ ft/sec

$s_0 = 64$ ft

(a) $$s(t) = -16t^2 + 16t + 64 = 0$$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $$v(t) = s'(t) = -32t + 16$$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

61. $a(t) = -9.8$

$$v(t) = \int -9.8\, dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \implies v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0)\, dt = -4.9t^2 + v_0t + C_2$$

$$f(0) = s_0 = C_2 \implies f(t) = -4.9t^2 + v_0t + C_2$$

62. From Exercise 61, $f(t) = -4.9t^2 + 1600$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1600$$

$$4.9t^2 = 1600$$

$$t^2 = \frac{1600}{4.9} \implies t \approx \sqrt{326.53} \approx 18.1 \text{ sec}$$

63. From Exercise 61, $f(t) = -4.9t^2 + 10t$.

$$v(t) = -9.8t + 10 = 0 \text{ (Maximum height when } v = 0.)$$

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 5.1 \text{ m}$$

64. From Exercise 61, $f(t) = -4.9t^2 + v_0t$. If

$$f(t) = 200 = -4.9t^2 + v_0t,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) = 200$$

$$\frac{-4.9\, v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 200$$

$$-4.9\, v_0^2 + 9.8\, v_0^2 = (9.8)^2 200$$

$$4.9\, v_0^2 = (9.8)^2\, 200$$

$$v_0^2 = 3920 \implies v_0 \approx 62.61 \text{ m/sec.}$$

65. $a = -1.6$

$v(t) = \int -1.6\,dt = -1.6t + v_0 = -1.6t$, since the stone was dropped, $v_0 = 0$.

$s(t) = \int (-1.6t)\,dt = -0.8t^2 + s_0$

$s(20) = 0 \implies -0.8(20)^2 + s_0 = 0$

$s_0 = 320$

Thus, the height of the cliff is 320 meters.

$v(t) = -1.6t$

$v(20) = -32$ m/sec

66. $\int v\,dv = -GM\int \frac{1}{y^2}\,dy$

$\frac{1}{2}v^2 = \frac{GM}{y} + C$

When $y = R$, $v = v_0$.

$\frac{1}{2}v_0^2 = \frac{GM}{R} + C$

$C = \frac{1}{2}v_0^2 - \frac{GM}{R}$

$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^2 - \frac{GM}{R}$

$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$

$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$

67. $x(t) = t^3 - 6t^2 + 9t - 2 \qquad 0 \le t \le 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$

$= 3(t^2 - 4t + 3) = 3(t-1)(t-3)$

$a(t) = v'(t) = 6t - 12 = 6(t-2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t-2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

68. $x(t) = (t-1)(t-3)^2 \qquad 0 \le t \le 5$

$= t^3 - 7t^2 + 15t - 9$

(a) $v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$

$a(t) = v'(t) = 6t - 14$

(b) $v(t) > 0$ when $0 < t < \frac{5}{3}$ and $3 < t < 5$.

(c) $a(t) = 6t - 14 = 0$ when $t = \frac{7}{3}$.

$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$

69. $v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \qquad t > 0$

$x(t) = \int v(t)\,dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \implies C = 2$

$x(t) = 2t^{1/2} + 2$

$a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}}$

70. (a) $a(t) = \cos t$

$$v(t) = \int a(t)\,dt = \int \cos t\,dt = \sin t + C_1 = \sin t \text{ (since } v_0 = 0)$$

$$f(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4$$

$$f(t) = -\cos t + 4$$

(b) $v(t) = 0 = \sin t$ for $t = k\pi,\ k = 0, 1, 2, \ldots$

71. (a) $v(0) = 25 \text{ km/hr} = 25 \cdot \dfrac{1000}{3600} = \dfrac{250}{36} \text{ m/sec}$

$$v(13) = 80 \text{ km/hr} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$$

$$a(t) = a \text{ (constant acceleration)}$$

$$v(t) = at + C$$

$$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$$

$$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$$

$$\frac{550}{36} = 13a$$

$$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$$

(b) $s(t) = a\dfrac{t^2}{2} + \dfrac{250}{36}t \quad (s(0) = 0)$

$$s(13) = \frac{275}{234}\frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$$

72. $v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$

$30 \text{ mph} = 44 \text{ ft/sec}$

$15 \text{ mph} = 22 \text{ ft/sec}$

$$a(t) = -a$$

$$v(t) = -at + 66$$

$$s(t) = -\frac{a}{2}t^2 + 66t \text{ (Let } s(0) = 0.)$$

$v(t) = 0$ after car moves 132 ft.

$-at + 66 = 0$ when $t = \dfrac{66}{a}$.

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$$= 132 \text{ when } a = \frac{33}{2} = 16.5.$$

$$a(t) = -16.5$$

$$v(t) = -16.5t + 66$$

$$s(t) = -8.25t^2 + 66t$$

(a) $-16.5t + 66 = 44$

$$t = \frac{22}{16.5} \approx 1.333$$

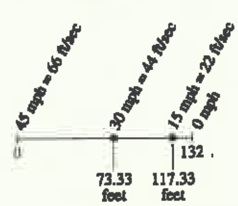
$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

(b) $-16.5t + 66 = 22$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$

(c)



It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

73. Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$$30t = 3t^2$$

$$0 = 3t^2 - 30t$$

$$0 = 3t(t - 10) \text{ when } t = 10 \text{ sec.}$$

(a) $s(10) = 3(10)^2 = 300$ ft

(b) $v(10) = 6(10) = 60$ ft/sec ≈ 41 mph

74. No, car 2 will be ahead of car 1. If $v_1(t)$ and $v_2(t)$ are the respective velocities, then

$$\int_0^{30} |v_2(t)|\,dt > \int_0^{30} |v_1(t)|\,dt.$$

75. $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \implies k = \frac{160^2}{1.4}$$

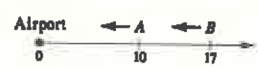
$$\approx 18{,}285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

76. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.

$v_A(t) = k_A t - 150$ $\qquad v_B = k_B t - 250$

$s_A(t) = \frac{1}{2}k_A t^2 - 150t + 10$ $\qquad s_B = \frac{1}{2}k_B t^2 - 250t + 17$



(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \implies k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}.$$

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625$$

—CONTINUED—

76. —CONTINUED—

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \implies k_B = \frac{135}{t_B}$$

$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

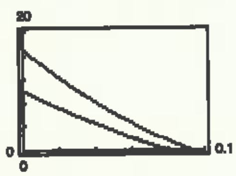
$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

$$t_B = \frac{34}{365}.$$

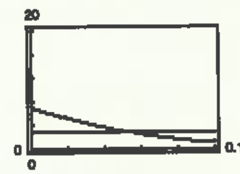
$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49{,}275}{34}$$

(b)



(c) $d = s_B(t) - s_A(t)$

Yes, $d < 3$ for $t > 0.0505$.



77. $\dfrac{dC}{dx} = 2x - 12$

$C(0) = 50$

$C(x) = x^2 - 12x + C$

$C(0) = 0 + C = 50 \implies C = 50$

Cost: $C(x) = x^2 - 12x + 50$

Average cost: $\overline{C}(x) = \dfrac{C(x)}{x} = x - 12 + \dfrac{50}{x}$

78. $\dfrac{dC}{dx} = \dfrac{\sqrt[4]{x}}{10} + 10$

$C(0) = 2300$

$$C(x) = \int\left(\frac{1}{10}x^{1/4} + 10\right)dx$$

$$= \frac{2}{25}x^{5/4} + 10x + C$$

$C(0) = 0 + 0 + C = 2300 \implies C = 2300$

Cost: $C(x) = \dfrac{2}{25}x^{5/4} + 10x + 2300$

Average cost: $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{2}{25}x^{1/4} + 10 + \dfrac{2300}{x}$

79. $\dfrac{dR}{dx} = 100 - 5x$

$R = 100x - \dfrac{5}{2}x^2 + C$

($C = 0$; if no units are sold, no revenue is generated.)

Revenue: $R = 100x - \dfrac{5}{2}x^2$

$$R = xp = x\left(100 - \frac{5}{2}x\right)$$

Demand: $p = 100 - \dfrac{5}{2}x$

80. $\dfrac{dR}{dx} = 100 - 6x - 2x^2$

$R = 100x - 3x^2 - \dfrac{2}{3}x^3 + C$

($C = 0$; if no units are sold, no revenue is generated.)

Revenue: $R = 100x - 3x^2 - \dfrac{2}{3}x^3$

$$R = xp = x\left(100 - 3x - \frac{2}{3}x^2\right)$$

Demand: $p = 100 - 3x - \dfrac{2}{3}x^2$

81. True

82. True

83. True

84. True

85. $f(0) = -4$. Graph of f' is given.

(a) $f'(4) \approx -1.0$

(b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than 4 units on $[0, 4]$.

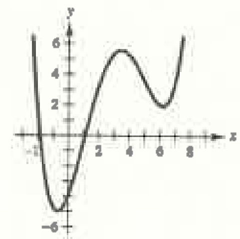
(c) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.

(d) f is an maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the first derivative test.

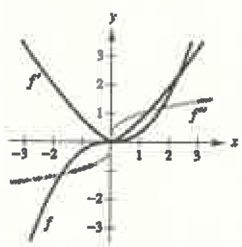
(e) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) f'' is a minimum at $x = 3$.

(g)



86. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



87. Let d be the distance traversed and a be the uniform acceleration. Furthermore, note that $v(0) = 0$ and let $s(0) = 0$.

$$a(t) = a$$

$$v(t) = at$$

$$s(t) = \frac{at^2}{2} = d \text{ when } t = \sqrt{\frac{2d}{a}}.$$

The highest speed is

$$v\left(\sqrt{\frac{2d}{a}}\right) = a\sqrt{\frac{2d}{a}} = \sqrt{2ad}$$

and the mean speed is

$$\frac{\sqrt{2ad}}{2} = \sqrt{\frac{ad}{2}}.$$

The time necessary to traverse the distance at the mean speed must satisfy the equation

$$\sqrt{\frac{ad}{2}}\,t = d \text{ or } t = \sqrt{\frac{2d}{a}}.$$

This is the same time traversed under uniform acceleration.

88. $$\frac{d}{dx}\left[[s(x)]^2 + [c(x)]^2\right] = 2s(x)s'(x) + 2c(x)c'(x)$$
$$= 2s(x)c(x) - 2c(x)s(x)$$
$$= 0$$

Thus,

$$[s(x)]^2 + [c(x)]^2 = k$$

for some constant k. Since,

$s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore,

$$[s(x)]^2 + [c(x)]^2 = 1.$$

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

Section 4.2 Area

1. $\displaystyle\sum_{i=1}^{5}(2i+1) = 2\sum_{i=1}^{5} i + \sum_{i=1}^{5} 1 = 2(1+2+3+4+5)+5 = 35$

2. $\displaystyle\sum_{k=2}^{5}(k+1)(k-3) = (3)(-1)+(4)(0)+(5)(1)+(6)(2) = 14$

3. $\displaystyle\sum_{k=0}^{4}\frac{1}{k^2+1} = 1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17} = \frac{158}{85}$

4. $\displaystyle\sum_{j=3}^{5}\frac{1}{j} = \frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{47}{60}$

5. $\displaystyle\sum_{k=1}^{4} c = c+c+c+c = 4c$

6. $\displaystyle\sum_{i=1}^{4}[(i-1)^2+(i+1)^3] = (0+8)+(1+27)+(4+64)+(9+125) = 238$

7. $\displaystyle\sum_{i=1}^{9}\frac{1}{3i}$

8. $\displaystyle\sum_{i=1}^{15}\frac{5}{1+i}$

9. $\displaystyle\sum_{j=1}^{8}\left[2\left(\frac{j}{8}\right)+3\right]$

10. $\displaystyle\sum_{j=1}^{4}\left[1-\left(\frac{j}{4}\right)^2\right]$

11. $\displaystyle\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3-\left(\frac{2i}{n}\right)\right]$

12. $\displaystyle\frac{2}{n}\sum_{i=1}^{n}\left[1-\left(\frac{2i}{n}-1\right)^2\right]$

13. $\displaystyle\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1+\frac{3i}{n}\right)^2\right]$

14. $\displaystyle\frac{1}{n}\sum_{i=0}^{n-1}\sqrt{1-\left(\frac{i}{n}\right)^2}$

15. $\displaystyle\sum_{i=1}^{20}2i = 2\sum_{i=1}^{20}i = 2\left[\frac{20(21)}{2}\right] = 420$

16. $\displaystyle\sum_{i=1}^{15}(2i-3) = 2\sum_{i=1}^{15}i - 3(15)$

$\displaystyle= 2\left[\frac{15(16)}{2}\right] - 45 = 195$

17. $\displaystyle\sum_{i=1}^{20}(i-1)^2 = \sum_{i=1}^{19}i^2$

$\displaystyle= \left[\frac{19(20)(39)}{6}\right] = 2470$

18. $\displaystyle\sum_{i=1}^{10}(i^2-1) = \sum_{i=1}^{10}i^2 - \sum_{i=1}^{10}1$

$\displaystyle= \left[\frac{10(11)(21)}{6}\right] - 10 = 375$

19. $\displaystyle\sum_{i=1}^{15}i(i-1)^2 = \sum_{i=1}^{15}i^3 - 2\sum_{i=1}^{15}i^2 + \sum_{i=1}^{15}i$

$\displaystyle= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2}$

$= 14{,}400 - 2{,}480 + 120$

$= 12{,}040$

20. $\displaystyle\sum_{i=1}^{10}i(i^2+1) = \sum_{i=1}^{10}i^3 + \sum_{i=1}^{10}i$

$\displaystyle= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2}\right] = 3080$

21. sum seq($x\boxed{\wedge}2+3, x, 1, 20, 1$) = 2930 (*TI-82*)

$\displaystyle\sum_{i=1}^{20}(i^2+3) = \frac{20(20+1)(2(20)+1)}{6} + 3(20)$

$\displaystyle= \frac{(20)(21)(41)}{6} + 60 = 2930$

22. sum seq($x\boxed{\wedge}3-2x, x, 1, 15, 1$) = 14,160 (*TI-82*)

$\displaystyle\sum_{i=1}^{15}(i^3-2i) = \frac{(15)^2(15+1)^2}{4} - 2\frac{15(15+1)}{2}$

$\displaystyle= \frac{(15)^2(16)^2}{4} - 15(16) = 14{,}160$

23. $\lim_{n\to\infty}\left[\left(\frac{4}{3n^3}\right)(2n^3+3n^2+n)\right] = \lim_{n\to\infty}\left[\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\right] = \frac{8}{3}$

24. $\lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{4}{3n^2}\right) = \frac{8}{3}$

25. $\lim_{n\to\infty}\left[\left(\frac{81}{n^4}\right)\frac{n^2(n+1)^2}{4}\right] = \frac{81}{4}\lim_{n\to\infty}\left[\frac{n^4+2n^3+n^2}{n^4}\right] = \frac{81}{4}(1) = \frac{81}{4}$

26. $\lim_{n\to\infty}\left[\left(\frac{64}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right] = \frac{64}{6}\lim_{n\to\infty}\left[\frac{2n^3+3n^2+n}{n^3}\right] = \frac{64}{6}(2) = \frac{64}{3}$

27. $\lim_{n\to\infty}\left[\left(\frac{18}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{18}{2}\lim_{n\to\infty}\left[\frac{n^2+n}{n^2}\right] = \frac{18}{2}(1) = 9$

28. $\lim_{n\to\infty}\left[\left(\frac{1}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{1}{2}\lim_{n\to\infty}\left[\frac{n^2+n}{n^2}\right] = \frac{1}{2}(1) = \frac{1}{2}$

29. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i}{n^2}\right) = \lim_{n\to\infty}\frac{16}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[8\left(\frac{n^2+n}{n^2}\right)\right] = 8\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 8$

30. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\frac{4}{2}\left(1+\frac{1}{n}\right) = 2$

31. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$$= \lim_{n\to\infty}\frac{1}{6}\left[\frac{2n^3-3n^2+n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2-(3/n)+(1/n^2)}{1}\right)\right] = \frac{1}{3}$$

32. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n^3}\sum_{i=1}^{n}(n+2i)^2$

$$= \lim_{n\to\infty}\frac{2}{n^3}\left[\sum_{i=1}^{n} n^2 + 4n\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\frac{2}{n^3}\left[n^3+(4n)\left(\frac{n(n+1)}{2}\right)+\frac{4(n)(n+1)(2n+1)}{6}\right]$$

$$= 2\lim_{n\to\infty}\left[1+2+\frac{2}{n}+\frac{4}{3}+\frac{2}{n}+\frac{2}{3n^2}\right]$$

$$= 2\left(1+2+\frac{4}{3}\right) = \frac{26}{3}$$

33. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}1+\frac{1}{n}\sum_{i=1}^{n} i\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1+\frac{n^2+n}{2n^2}\right] = 2\left(1+\frac{1}{2}\right) = 3$

34. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n+2i)^3$

$$= 2\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n^3+6n^2i+12ni^2+8i^3)$$

$$= 2\lim_{n\to\infty}\frac{1}{n^4}\left[n^4+6n^2\left(\frac{n(n+1)}{2}\right)+12n\left(\frac{n(n+1)(2n+1)}{6}\right)+8\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= 2\lim_{n\to\infty}\left(1+3+\frac{3}{n}+4+\frac{6}{n}+\frac{2}{n^2}+2+\frac{4}{n}+\frac{2}{n^2}\right)$$

$$= 2\lim_{n\to\infty}\left(10+\frac{13}{n}+\frac{4}{n^2}\right) = 20$$

35. $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \dfrac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \dfrac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$

36. $S(8) = \left(\sqrt{\frac{1}{4}} + 1\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 1\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 1\right)\frac{1}{4} + (\sqrt{1} + 1)\frac{1}{4}$

$+ \left(\sqrt{\frac{5}{4}} + 1\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 1\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 1\right)\frac{1}{4} + (\sqrt{2} + 1)\frac{1}{4}$

$= \frac{1}{4}\left(8 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 4.038$

$s(8) = (0 + 1)\frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 1\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 1\right)\frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 1\right)\frac{1}{4} \approx 3.685$

37. $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

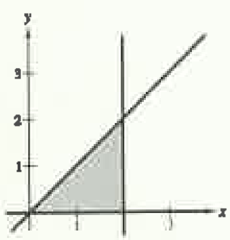
$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

38. $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$

$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

39. (a)



(b) $\Delta x = \dfrac{2 - 0}{n} = \dfrac{2}{n}$

Endpoints:

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \cdots < (n - 1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$

$= \sum_{i=1}^{n} f\left(\frac{2i - 2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$

(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $\lim_{n\to\infty} \sum_{i=1}^{n}\left[(i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^{n}(i - 1)$

$= \lim_{n\to\infty} \frac{4}{n^2}\left[\frac{n(n + 1)}{2} - n\right]$

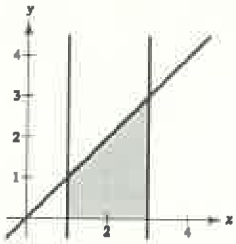
$= \lim_{n\to\infty} \left[\frac{2(n + 1)}{n} - \frac{4}{n}\right] = 2$

$\lim_{n\to\infty} \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^{n} i$

$= \lim_{n\to\infty} \left(\frac{4}{n^2}\right)\frac{n(n + 1)}{2}$

$= \lim_{n\to\infty} \frac{2(n + 1)}{n} = 2$

40. (a)



(b) $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \cdots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

(f) $$\lim_{n\to\infty} \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n + \frac{2}{n}\left(\frac{n(n+1)}{2} - n\right)\right]$$

$$= \lim_{n\to\infty}\left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n\to\infty}\left[4 - \frac{2}{n}\right] = 4$$

$$\lim_{n\to\infty} \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\left[n + \left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

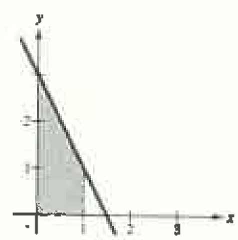
$$= \lim_{n\to\infty}\left[2 + \frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4 + \frac{2}{n}\right] = 4$$

41. $y = -2x + 3$ on $[0, 1]$. $\left(\text{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-2\left(\frac{i}{n}\right) + 3\right]\left(\frac{1}{n}\right)$$

$$= 3 - \frac{2}{n^2}\sum_{i=1}^{n} i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 2$$



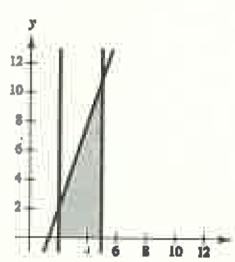
42. $y = 3x - 4$ on $[2, 5]$. $\left(\text{Note: } \Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

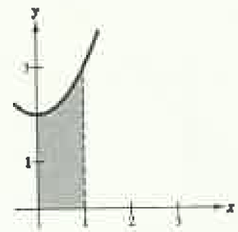
$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^{n} i - 12$$

$$= 6 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$



43. $y = x^2 + 2$ on $[0, 1]$. $\left(\text{Note: } \Delta x = \dfrac{1}{n}\right)$

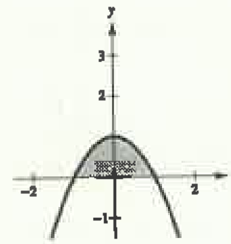


$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right)$$

$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{7}{3}$$

44. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. $\left(\text{Note: } \Delta x = \dfrac{1}{n}\right)$



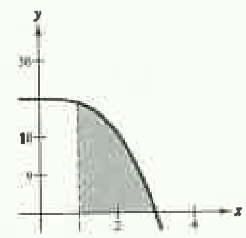
$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\frac{1}{2}\text{Area} = \lim_{n\to\infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

45. $y = 27 - x^3$ on $[1, 3]$. $\left(\text{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[27 - \left(1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(26 - \frac{6i}{n} - \frac{12i^2}{n^2} - \frac{8i^3}{n^3}\right)$$

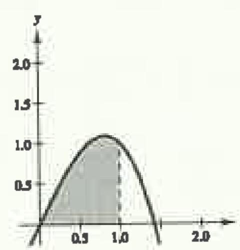
$$= \frac{2}{n}\left[26n - \frac{6}{n}\left(\frac{n(n+1)}{2}\right) - \frac{12}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{8}{n^3}\left(\frac{n^2(n+1)^2}{4}\right)\right]$$

$$= 34 - \frac{26}{n} - \frac{8}{n^2}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 34$$

46. $y = 2x - x^3$ on $[0, 1]$. $\left(\text{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.



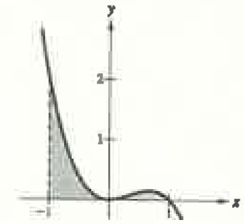
$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$

47. $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}\right)$



Again, $T(n)$ is neither an upper nor a lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right)$$

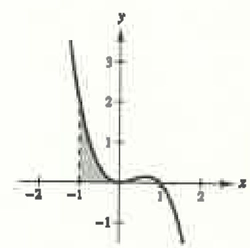
$$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

48. $y = x^2 - x^3$ on $[-1, 0]$. $\left(\textbf{Note: } \Delta x = \dfrac{0-(-1)}{n} = \dfrac{1}{n}\right)$



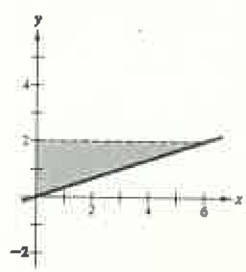
$$s(n) = \sum_{i=1}^{n} f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\left(\frac{1}{n}\right)\right] = 2 - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$

49. $f(y) = 3y$ on $[0, 2]$. $\left(\textbf{Note: } \Delta y = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

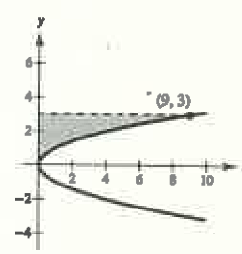


$$S(n) = \sum_{i=1}^{n} f(m_i)\,\Delta y = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2}\sum_{i=1}^{n} i = \left(\frac{12}{n^2}\right)\cdot\frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(6 + \frac{6}{n}\right) = 6$$

50. $f(y) = y^2$ on $[0, 3]$. $\left(\textbf{Note: } \Delta y = \dfrac{3-0}{n} = \dfrac{3}{n}\right)$



$$S(n) = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(\frac{3i}{n}\right)^2\left(\frac{3}{n}\right) = \frac{27}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2}\left(\frac{2n^2 + 3n + 1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$

51. $f(x) = x^2 + 3, 0 \le x \le 2, n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{1}{2}, c_1 = \frac{1}{4}, c_2 = \frac{3}{4}, c_3 = \frac{5}{4}, c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}[c_i^2 + 3]\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right)\right]$$

$$= \frac{69}{8}$$

52. $f(x) = x^2 + 4x, 0 \le x \le 4, n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = 1, c_1 = \frac{1}{2}, c_2 = \frac{3}{2}, c_3 = \frac{5}{2}, c_4 = \frac{7}{2}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}[c_i^2 + 4c_i](1)$$

$$= \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right]$$

$$= 53$$

53. $f(x) = \tan x, 0 \le x \le \dfrac{\pi}{4}, n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{16}, c_1 = \frac{\pi}{32}, c_2 = \frac{3\pi}{32}, c_3 = \frac{5\pi}{32}, c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

54. $f(x) = \sin x, 0 \le x \le \dfrac{\pi}{2}, n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{8}, c_1 = \frac{\pi}{16}, c_2 = \frac{3\pi}{16}, c_3 = \frac{5\pi}{16}, c_4 = \frac{7\pi}{16}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}(\sin c_i)\left(\frac{\pi}{8}\right)$$

$$= \frac{\pi}{8}\left(\sin\frac{\pi}{16} + \sin\frac{3\pi}{16} + \sin\frac{5\pi}{16} + \sin\frac{7\pi}{16}\right) \approx 1.006$$

55. $f(x) = \sqrt{x}$ on $[0, 4]$.

n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is 16/3)

56. $f(x) = \dfrac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

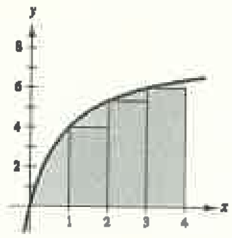
57. $f(x) = \tan\left(\dfrac{\pi x}{8}\right)$ on $[1, 3]$.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

58. $f(x) = \cos\sqrt{x}$ on $[0, 2]$.

n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

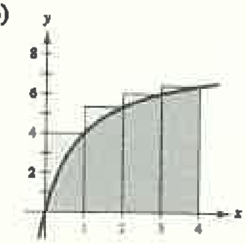
59. (a)



Lower sum:

$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

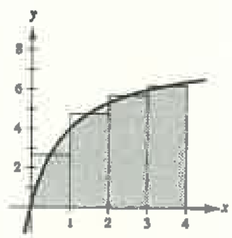
(b)



Upper sum:

$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

(c)



Midpoint Rule:

$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

(d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

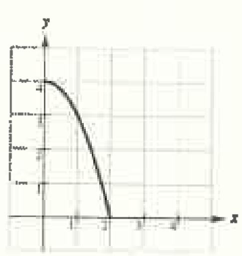
(f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

60. $f(x) = \sqrt[3]{x}, 0 \le x \le 8$

n	10	20	50	100	200
$s(n)$	10.998	11.519	11.816	11.910	11.956
$S(n)$	12.598	12.319	12.136	12.070	12.036
$M(n)$	12.040	12.016	12.005	12.002	12.001

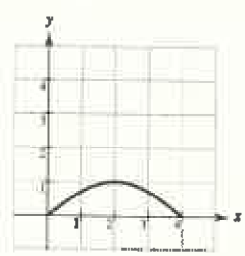
(**Note:** exact answer is 12.)

61.



b. $A \approx 6$ square units

62.

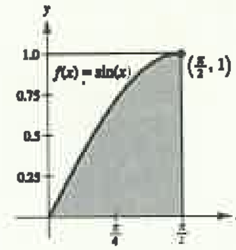


a. $A \approx 3$ square units

63. True. (Theorem 4.2 (2))

64. True. (Theorem 4.3)

65. $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$

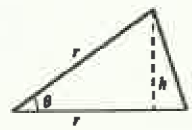


Let A_1 = area bounded by $f(x) = \sin x$, the x-axis, $x = 0$ and $x = \pi/2$. Let A_2 = area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$. In this program, the computer is generating N_2 pairs of random points in the rectangle whose area is represented by A_2. It is keeping track of how many of these points, N_1, lie in the region whose area is represented by A_1. Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2}A_2.$$

The larger N_2 is the better the approximation to A_1.

66. (a) $\theta = \dfrac{2\pi}{n}$



(b) $\sin\theta = \dfrac{h}{r}$

$$h = r\sin\theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

(c) $A_n = n\left(\dfrac{1}{2}r^2\sin\dfrac{2\pi}{n}\right) = \dfrac{r^2n}{2}\sin\dfrac{2\pi}{n} = \pi r^2\left(\dfrac{\sin(2\pi/n)}{2\pi/n}\right)$

Let $x = 2\pi/n$. As $n \to \infty$, $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0} \pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

67. Suppose there are n rows in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \tfrac{1}{2}(n)(n + 1).$$

68. (a) $(i + 1)^3 - i^3 = (i^3 + 3i^2 + 3i + 1) - i^3 = 3i^2 + 3i + 1$

(b) $\displaystyle\sum_{i=1}^{n}(3i^2 + 3i + 1) = \sum_{i=1}^{n}[(i + 1)^3 - i^3]$

$$= (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + (n^3 - (n - 1)^3) + ((n + 1)^3 - n^3)$$

$$= -1 + (n + 1)^3$$

(c) $\displaystyle\sum_{i=1}^{n}(3i^2 + 3i + 1) = 3\sum_{i=1}^{n}i^2 + 3\sum_{i=1}^{n}i + \sum_{i=1}^{n}1$

$$-1 + (n + 1)^3 = 3\sum_{i=1}^{n}i^2 + \frac{3n(n + 1)}{2} + n$$

$$n^3 + 3n^2 + 3n = 3\sum_{i=1}^{n}i^2 + \frac{3}{2}n^2 + \frac{5}{2}n$$

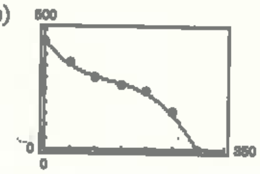
$$\frac{n^3 + (3/2)n^2 + (1/2)n}{3} = \sum_{i=1}^{n}i^2$$

$$\frac{2n^3 + 3n^2 + n}{6} = \sum_{i=1}^{n}i^2$$

$$\frac{n(2n + 1)(n + 1)}{6} = \sum_{i=1}^{n}i^2$$

69. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)



(c) Using the integration capability of a graphing utility, you obtain

$A \approx 76{,}897.5 \text{ ft}^2$.

Section 4.3 Riemann Sums and Definite Integrals

1. $\int_0^5 3\,dx$

2. $\int_0^2 (4 - 2x)\,dx$

3. $\int_{-4}^4 (4 - |x|)\,dx$

4. $\int_0^2 x^2\,dx$

5. $\int_{-2}^2 (4 - x^2)\,dx$

6. $\int_{-1}^1 \frac{1}{x^2 + 1}\,dx$

7. $\int_0^\pi \sin x\,dx$

8. $\int_0^{\pi/4} \tan x\,dx$

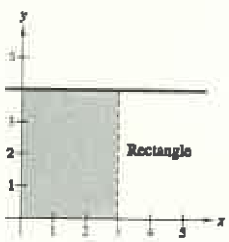
9. $\int_0^2 y^3\,dy$

10. $\int_0^2 (y - 2)^2\,dy$

11. Rectangle

$A = bh = 3(4)$

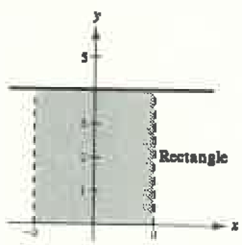
$A = \int_0^3 4\,dx = 12$



12. Rectangle

$A = bh = 2(4)(a)$

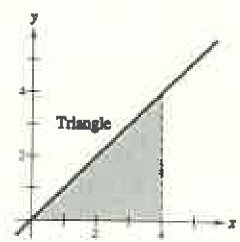
$A = \int_{-a}^a 4\,dx = 8a$



13. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$

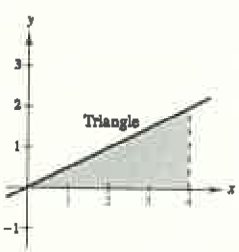
$A = \int_0^4 x\,dx = 8$



14. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$

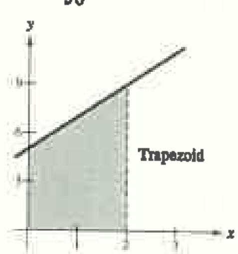
$A = \int_0^4 \frac{x}{2}\,dx = 4$



15. Trapezoid

$A = \frac{b_1 + b_2}{2}h = \left(\frac{5 + 9}{2}\right)2$

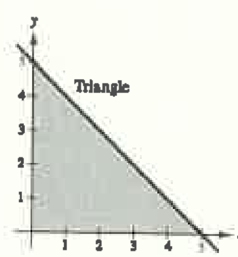
$A = \int_0^2 (2x + 5)\,dx = 14$



16. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(5)(5)$

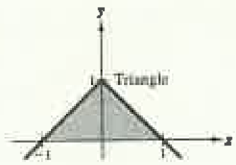
$A = \int_0^5 (5 - x)\,dx = \frac{25}{2}$



17. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$

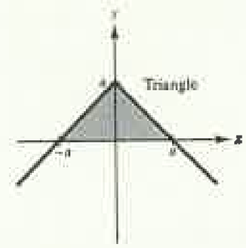
$A = \int_{-1}^{1} (1 - |x|)\,dx = 1$



18. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$

$A = \int_{-a}^{a} (a - |x|)\,dx = a^2$



19. Semicircle

$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$

$A = \int_{-3}^{3} \sqrt{9 - x^2}\,dx = \frac{9\pi}{2}$

20. Semicircle

$A = \frac{1}{2}\pi r^2$

$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2$

21. (a) $\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\,dx = -\int_0^5 f(x)\,dx = -10$

(c) $\int_5^5 f(x)\,dx = 0$

(d) $\int_0^5 3f(x)\,dx = 3\int_0^5 f(x)\,dx = 3(10) = 30$

22. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

23. (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx$

$= 10 + (-2) = 8$

(b) $\int_2^6 [g(x) - f(x)]\,dx = \int_2^6 g(x)\,dx - \int_2^6 f(x)\,dx$

$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\,dx = 2\int_2^6 g(x)\,dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\,dx = 3\int_2^6 f(x)\,dx = 3(10) = 30$

24. (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx = 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

25. $y = 6$ on $[4, 10]$. $\left(\text{Note: } \Delta x = \frac{10 - 4}{n} = \frac{6}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(4 + \frac{6i}{n}\right)\left(\frac{6}{n}\right) = \sum_{i=1}^{n} 6\left(\frac{6}{n}\right) = \sum_{i=1}^{n} \frac{36}{n} = 36$$

$$\int_4^{10} 6\,dx = \lim_{n \to \infty} 36 = 36$$

26. $y = x$ on $[-2, 3]$. $\left(\text{Note: } \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i$$

$$= -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

27. $y = x^3$ on $[-1, 1]$. $\left(\text{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty}\frac{2}{n} = 0$$

28. $y = x^3$ on $[0, 1]$. $\left(\text{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\int_{0}^{1} x^3\,dx = \frac{1}{4}\lim_{n\to\infty}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{1}{4}$$

29. $y = x^2 + 1$ on $[1, 2]$. $\left(\text{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_{1}^{2}(x^2 + 1)\,dx = \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

30. $y = 4x^2$ on $[1, 2]$. $\left(\text{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[4\left(1 + \frac{i}{2}\right)^2\right]\left(\frac{1}{n}\right) = 4\sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2}\right]\left(\frac{1}{n}\right)$$

$$= 4\left[1 + \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2\right] = 4\left[1 + \left(\frac{2}{n^2}\right)\frac{n(n+1)}{2} + \left(\frac{1}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 4\left[1 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = 4\left(\frac{7}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right)$$

$$\int_{1}^{2} 4x^2\,dx = 4\lim_{n\to\infty}\left(\frac{7}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{28}{3}$$

31. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (3c_i + 10)\,\Delta x_i = \int_{-1}^{5} (3x + 10)\,dx$

on the interval $[-1, 5]$.

32. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i = \int_{0}^{4} 6x(4 - x)^2\,dx$

on the interval $[0, 4]$.

33. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{c_i^2 + 4}\,\Delta x_i = \int_{0}^{3} \sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

34. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(\frac{3}{c_i^2}\right)\Delta x_i = \int_{1}^{3} \frac{3}{x^2}\,dx$

on the interval $[1, 3]$.

35. $\int_{0}^{3} x\sqrt{3 - x}\,dx$

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

36. $\int_{0}^{3} \frac{5}{x^2 + 1}\,dx$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	7.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

37. $\int_{0}^{\pi/2} \sin^2 x\,dx$

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

38. $\int_{0}^{3} \sin x\,dx$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

39. The left endpoint approximation will be greater than the actual area: $>$

40. The right endpoint approximation will be less than the actual area: $<$

41. Because the curve is concave upward, the midpoint approximation will be less than the actual area: $<$

42. The average of Exercise 39 and Exercise 40 consists of a trapezoidal approximation, and is greater than the exact area: $>$

43. (a) Quarter circle below x-axis: $-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x-axis: $-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

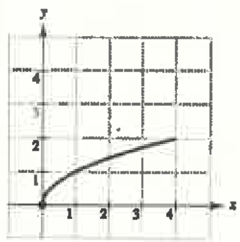
(e) Sum of absolute values of (b) and (c): $4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answer to (d) plus $2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

44. (a) $\displaystyle\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx = 4 + 10 = 14$ (b) $\displaystyle\int_{-2}^3 f(x + 2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

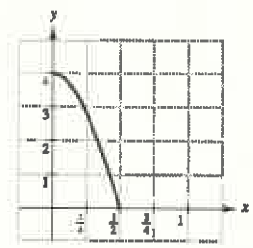
(c) $\displaystyle\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even) (d) $\displaystyle\int_{-5}^5 f(x)\,dx = 0$ (f odd)

45.



a. $A \approx 5$ square units

46.



b. $A \approx \frac{4}{3}$ square units

47. True

48. False

$$\int_0^1 x\sqrt{x}\,dx \neq \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{x}\,dx\right)$$

49. True

50. True

51. False

$$\int_0^2 (-x)\,dx = -2$$

52. False

$$\int_{-2}^4 x\,dx = 6$$

53. $f(x) = x^2 + 3x$, $[0, 8]$

$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$

$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$

$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$

$$\begin{aligned}\sum_{i=1}^4 f(c_i)\,\Delta x &= f(1)\,\Delta x_1 + f(2)\,\Delta x_2 + f(5)\,\Delta x_3 + f(8)\,\Delta x_4\\ &= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272\end{aligned}$$

54. $f(x) = \sin x$, $[0, 2\pi]$

$$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$$

$$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$$

$$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

55. $f(x) = \sqrt{x}, y = 0, x = 0, x = 2, c_i = \dfrac{2i^2}{n^2}$

$$\Delta x_i = \frac{2i^2}{n^2} - \frac{2(i-1)^2}{n^2} = \frac{2(2i-1)}{n^2}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{2i^2}{n^2}}\left[\frac{2(2i-1)}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{2\sqrt{2}}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n\to\infty} \frac{2\sqrt{2}}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{2\sqrt{2}}{n^3}\left[\frac{4n^3 + 3n^2 - n}{6}\right] = \lim_{n\to\infty} \sqrt{2}\left[\frac{4}{3} + \frac{1}{n} - \frac{2}{n^2}\right] = \frac{4\sqrt{2}}{3}$$

56. $f(x) = \sqrt[3]{x}, y = 0, x = 0, x = 1, c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} (3i^3 - 3i^2 + i)$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}$$

57. $f(x) = \dfrac{1}{x-4}$

is not integrable on the interval $[3, 5]$ and f has a discontinuity at $x = 4$.

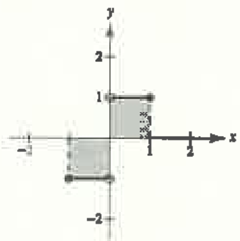
58. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

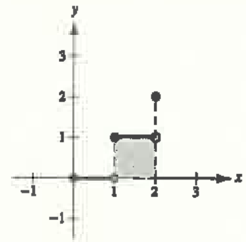
59. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^{1} \frac{|x|}{x}\,dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x-axis for $-1 \le x \le 1$. You see that the integral equals 0.



60. To find $\int_0^2 [\![x]\!]\,dx$, use a geometric approach.



Thus,

$$\int_0^2 [\![x]\!]\,dx = 1(2 - 1) = 1.$$

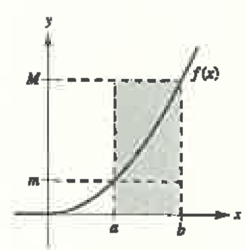
61. $$\lim_{n\to\infty} \frac{1}{n^3}[1^2 + 2^2 + 3^2 + \cdots + n^2] = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

62. From the graph, the area of the rectangle with width m and length $b - a$ is $m(b - a)$, which is less than the area given by $\int_a^b f(x)\,dx$. The area of the rectangle with width M and length $b - a$ is $M(b - a)$, which greater than the area given by $\int_a^b f(x)\,dx$. Thus,



$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

$$1 \le \sqrt{1 + x^4} \le \sqrt{2} \text{ on } [0, 1]$$

$$\int_0^1 dx \le \int_0^1 \sqrt{1 + x^4}\,dx \le \int_0^1 \sqrt{2}\,dx$$

$$1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}.$$

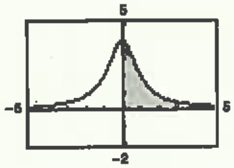
63. Since $-|f(x)| \le f(x) \le |f(x)|$,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Section 4.4 The Fundamental Theorem of Calculus

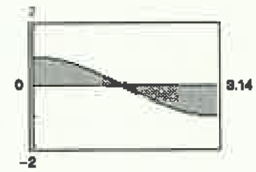
1. $f(x) = \dfrac{4}{x^2+1}$

$\displaystyle\int_0^{\pi} \frac{4}{x^2+1}\,dx$ is positive.



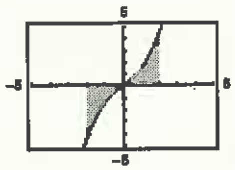
2. $f(x) = \cos x$

$\displaystyle\int_0^{\pi} \cos x\,dx = 0$



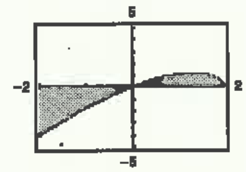
3. $f(x) = x\sqrt{x^2+1}$

$\displaystyle\int_{-2}^{2} x\sqrt{x^2+1}\,dx = 0$



4. $f(x) = x\sqrt{2-x}$

$\displaystyle\int_{-2}^{2} x\sqrt{2-x}\,dx$ is negative.



5. $\displaystyle\int_0^1 2x\,dx = \Big[x^2\Big]_0^1 = 1 - 0 = 1$

6. $\displaystyle\int_2^7 3\,dv = \Big[3v\Big]_2^7 = 3(7) - 3(2) = 15$

7. $\displaystyle\int_{-1}^{0} (x-2)\,dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^{0} = 0 - \left(\frac{1}{2} + 2\right) = -\frac{5}{2}$

8. $\displaystyle\int_2^5 (-3v+4)\,dv = \left[-\frac{3}{2}v^2 + 4v\right]_2^5 = \left(-\frac{75}{2} + 20\right) - (-6+8) = -\frac{39}{2}$

9. $\displaystyle\int_{-1}^{1} (t^2-2)\,dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^{1} = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$

10. $\displaystyle\int_0^3 (3x^2 + x - 2)\,dx = \left[x^3 + \frac{x^2}{2} - 2x\right]_0^3 = \left(27 + \frac{9}{2} - 6\right) - 0 = \frac{51}{2}$

11. $\displaystyle\int_0^1 (2t-1)^2\,dt = \int_0^1 (4t^2 - 4t + 1)\,dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$

12. $\displaystyle\int_{-1}^{1} (t^3 - 9t)\,dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2\right]_{-1}^{1} = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = 0$

13. $\displaystyle\int_1^2 \left(\frac{3}{x^2} - 1\right)dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3-1) = \frac{1}{2}$

14. $\displaystyle\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right)du = \left[\frac{u^2}{2} + \frac{1}{u}\right]_{-2}^{-1} = \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$

15. $\displaystyle\int_1^4 \frac{u-2}{\sqrt{u}}\,du = \int_1^4 (u^{1/2} - 2u^{-1/2})\,du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}(\sqrt{4})^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$

16. $\displaystyle\int_{-3}^{3} v^{1/3}\,dv = \left[\frac{3}{4}v^{4/3}\right]_{-3}^{3} = \frac{3}{4}[(\sqrt[3]{-3})^4] - (\sqrt[3]{-3})^4] = 0$

17. $\displaystyle\int_{-1}^{1}(\sqrt[3]{t}-2)\,dt=\left[\frac{3}{4}t^{4/3}-2t\right]_{-1}^{1}=\left(\frac{3}{4}-2\right)-\left(\frac{3}{4}+2\right)=-4$

18. $\displaystyle\int_{1}^{8}\sqrt{\frac{2}{x}}\,dx=\sqrt{2}\int_{1}^{8}x^{-1/2}\,dx=\left[\sqrt{2}(2)x^{1/2}\right]_{1}^{8}=\left[2\sqrt{2x}\right]_{1}^{8}=8-2\sqrt{2}$

19. $\displaystyle\int_{0}^{1}\frac{x-\sqrt{x}}{3}\,dx=\frac{1}{3}\int_{0}^{1}(x-x^{1/2})\,dx=\frac{1}{3}\left[\frac{x^2}{2}-\frac{2}{3}x^{3/2}\right]_{0}^{1}=\frac{1}{3}\left(\frac{1}{2}-\frac{2}{3}\right)=-\frac{1}{18}$

20. $\displaystyle\int_{0}^{2}(2-t)\sqrt{t}\,dt=\int_{0}^{2}(2t^{1/2}-t^{3/2})\,dt=\left[\frac{4}{3}t^{3/2}-\frac{2}{5}t^{5/2}\right]_{0}^{2}=\left[\frac{t\sqrt{t}}{15}(20-6t)\right]_{0}^{2}=\frac{2\sqrt{2}}{15}(20-12)=\frac{16\sqrt{2}}{15}$

21. $\displaystyle\int_{-1}^{0}(t^{1/3}-t^{2/3})\,dt=\left[\frac{3}{4}t^{4/3}-\frac{3}{5}t^{5/3}\right]_{-1}^{0}=0-\left(\frac{3}{4}+\frac{3}{5}\right)=-\frac{27}{20}$

22. $\displaystyle\int_{-8}^{-1}\frac{x-x^2}{2\sqrt[3]{x}}\,dx=\frac{1}{2}\int_{-8}^{-1}(x^{2/3}-x^{5/3})\,dx$

$\displaystyle=\frac{1}{2}\left[\frac{3}{5}x^{5/3}-\frac{3}{8}x^{8/3}\right]_{-8}^{-1}=\left[\frac{x^{5/3}}{80}(24-15x)\right]_{-8}^{-1}=-\frac{1}{80}(39)+\frac{32}{80}(144)=\frac{4569}{80}$

23. $\displaystyle\int_{0}^{3}|2x-3|\,dx=\int_{0}^{3/2}(3-2x)\,dx+\int_{3/2}^{3}(2x-3)\,dx\ \left(\text{split up the integral at the zero } x=\frac{3}{2}\right)$

$\displaystyle=\left[3x-x^2\right]_{0}^{3/2}+\left[x^2-3x\right]_{3/2}^{3}=\left(\frac{9}{2}-\frac{9}{4}\right)-0+(9-9)-\left(\frac{9}{4}-\frac{9}{2}\right)=2\left(\frac{9}{2}-\frac{9}{4}\right)=\frac{9}{2}$

24. $\displaystyle\int_{0}^{4}|x^2-4x+3|\,dx=\int_{0}^{1}(x^2-4x+3)\,dx-\int_{1}^{3}(x^2-4x+3)\,dx+\int_{3}^{4}(x^2-4x+3)\,dx$ (split up the integral at the zeros $x=1,3$)

$\displaystyle=\left[\frac{x^3}{3}-2x^2+3x\right]_{0}^{1}-\left[\frac{x^3}{3}-2x^2+3x\right]_{1}^{3}+\left[\frac{x^3}{3}-2x^2+3x\right]_{3}^{4}$

$\displaystyle=\left(\frac{1}{3}-2+3\right)-(9-18+9)+\left(\frac{1}{3}-2+3\right)+\left(\frac{64}{3}-32+12\right)-(9-18+9)$

$\displaystyle=\frac{4}{3}-0+\frac{4}{3}+\frac{4}{3}-0=4$

25. $\displaystyle\int_{0}^{\pi}(1+\sin x)\,dx=\left[x-\cos x\right]_{0}^{\pi}=(\pi+1)-(0-1)=2+\pi$

26. $\displaystyle\int_{0}^{\pi/4}\frac{1-\sin^2\theta}{\cos^2\theta}\,d\theta=\int_{0}^{\pi/4}d\theta=\left[\theta\right]_{0}^{\pi/4}=\frac{\pi}{4}$

27. $\displaystyle\int_{-\pi/6}^{\pi/6}\sec^2 x\,dx=\left[\tan x\right]_{-\pi/6}^{\pi/6}=\frac{\sqrt{3}}{3}-\left(-\frac{\sqrt{3}}{3}\right)=\frac{2\sqrt{3}}{3}$

28. $\displaystyle\int_{\pi/4}^{\pi/2}(2-\csc^2 x)\,dx=\left[2x+\cot x\right]_{\pi/4}^{\pi/2}=(\pi+0)-\left(\frac{\pi}{2}+1\right)=\frac{\pi}{2}-1=\frac{\pi-2}{2}$

29. $\displaystyle\int_{-\pi/3}^{\pi/3}4\sec\theta\tan\theta\,d\theta=\left[4\sec\theta\right]_{-\pi/3}^{\pi/3}=4(2)-4(2)=0$

30. $\displaystyle\int_{-\pi/2}^{\pi/2}(2t+\cos t)\,dt = \left[t^2+\sin t\right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4}+1\right)-\left(\frac{\pi^2}{4}-1\right) = 2$

31. $\displaystyle\int_0^3 10{,}000(t-6)\,dt = 10{,}000\left[\frac{t^2}{2}-6t\right]_0^3 = -\$135{,}000$

32. $\displaystyle P = \frac{2}{\pi}\int_0^{\pi/2}\sin\theta\,d\theta = \left[-\frac{2}{\pi}\cos\theta\right]_0^{\pi/2} = -\frac{2}{\pi}(0-1) = \frac{2}{\pi} \approx 63.7\%$

33. $\displaystyle A = \int_0^1 (x-x^2)\,dx = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

34. $\displaystyle A = \int_{-1}^1 (1-x^4)\,dx = \left[x-\frac{1}{5}x^5\right]_{-1}^1 = \frac{8}{5}$

35. $\displaystyle A = \int_0^3 (3-x)\sqrt{x}\,dx = \int_0^3 (3x^{1/2}-x^{3/2})\,dx = \left[2x^{3/2}-\frac{2}{5}x^{5/2}\right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10-2x)\right]_0^3 = \frac{12\sqrt{3}}{5}$

36. $\displaystyle A = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2}+1 = \frac{1}{2}$

37. $\displaystyle A = \int_0^{\pi/2}\cos x\,dx = \left[\sin x\right]_0^{\pi/2} = 1$

38. $\displaystyle A = \int_0^{\pi}(x+\sin x)\,dx = \left[\frac{x^2}{2}-\cos x\right]_0^{\pi} = \frac{\pi^2}{2}+2 = \frac{\pi^2+4}{2}$

39. Since $y \ge 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2+1)\,dx = \left[x^3+x\right]_0^2 = 8+2 = 10.$$

40. Since $y \ge 0$ on $[0, 4]$,

$$A = \int_0^4 \left(1+\sqrt{x}\right)dx = \left[x+\frac{2}{3}x^{3/2}\right]_0^4 = 4+\frac{2}{3}(8) = \frac{28}{3}.$$

41. Since $y \ge 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3+x)\,dx = \left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^2 = 4+2 = 6.$$

42. Since $y \ge 0$ on $[0, 3]$,

$$A = \int_0^3 (3x-x^2)\,dx = \left[\frac{3}{2}x^2-\frac{x^3}{3}\right]_0^3 = \frac{9}{2}.$$

43. $\displaystyle\int_0^2 \left(x-2\sqrt{x}\right)dx = \left[\frac{x^2}{2}-\frac{4x^{3/2}}{3}\right]_0^2 = 2-\frac{8\sqrt{2}}{3}$

$$f(c)(2-0) = \frac{6-8\sqrt{2}}{3}$$

$$c-2\sqrt{c} = \frac{3-4\sqrt{2}}{3}$$

$$c-2\sqrt{c}+1 = \frac{3-4\sqrt{2}}{3}+1$$

$$\left(\sqrt{c}-1\right)^2 = \frac{6-4\sqrt{2}}{3}$$

$$\sqrt{c}-1 = \pm\sqrt{\frac{6-4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6-4\sqrt{2}}{3}}\right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

44. $\displaystyle\int_1^3 \frac{9}{x^3}\,dx = \left[-\frac{9}{2x^2}\right]_1^3 = -\frac{1}{2}+\frac{9}{2} = 4$

$$f(c)(3-1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

45. $\displaystyle\int_{-\pi/4}^{\pi/4} 2\sec^2 x\,dx = \Big[2\tan x\Big]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$

$$f(c)\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 4$$

$$2\sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm\frac{2}{\sqrt{\pi}}$$

$$c = \pm\operatorname{arcsec}\left(\frac{2}{\sqrt{\pi}}\right)$$

$$= \pm\arccos\frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

46. $\displaystyle\int_{-\pi/3}^{\pi/3} \cos x\,dx = \Big[\sin x\Big]_{-\pi/3}^{\pi/3} = \sqrt{3}$

$$f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3}$$

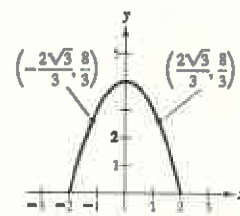
$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

$$c \approx \pm 0.5971$$

47. $\displaystyle\frac{1}{2-(-2)}\int_{-2}^{2}(4 - x^2)\,dx = \frac{1}{4}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{1}{4}\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] = \frac{8}{3}$

Average value $= \dfrac{8}{3}$

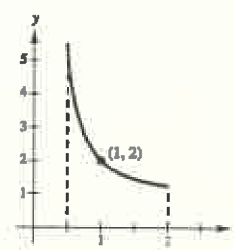
$4 - x^2 = \dfrac{8}{3}$ when $x^2 = 4 - \dfrac{8}{3}$ or $x = \pm\dfrac{2\sqrt{3}}{3} \approx \pm 1.155$.



48. $\displaystyle\frac{1}{2-(1/2)}\int_{1/2}^{2}\frac{x^2+1}{x^2}\,dx = \frac{2}{3}\int_{1/2}^{2}(1 + x^{-2})\,dx = \frac{2}{3}\left[x - \frac{1}{x}\right]_{1/2}^{2} = \frac{2}{3}\left(\frac{3}{2} + \frac{3}{2}\right) = 2$

Average value $= 2$

In the interval $\left[\dfrac{1}{2}, 2\right]$, $\dfrac{x^2+1}{x^2} = 2$ when $x^2 + 1 = 2x^2$ or $x = 1$.

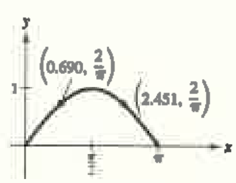


49. $\displaystyle\frac{1}{\pi - 0}\int_0^{\pi}\sin x\,dx = \left[-\frac{1}{\pi}\cos x\right]_0^{\pi} = \frac{2}{\pi}$

Average value $= \dfrac{2}{\pi}$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690, 2.451$$

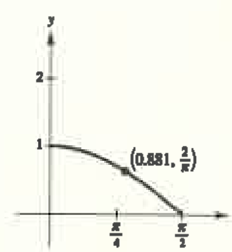


50. $\displaystyle\frac{1}{(\pi/2) - 0}\int_0^{\pi/2}\cos x\,dx = \left[\frac{2}{\pi}\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$

Average value $= \dfrac{2}{\pi}$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$



51. $\displaystyle\int_0^2 f(x)\,dx = -(\text{area of region } A) = -1.5$

52. $\displaystyle\int_2^6 f(x)\,dx = (\text{area or region } B) = \int_0^6 f(x)\,dx - \int_0^2 f(x)\,dx$

$$= 3.5 - (-1.5) = 5.0$$

53. $\displaystyle\int_0^6 |f(x)|\,dx = -\int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = 1.5 + 5.0 = 6.5$

54. $\displaystyle\int_0^2 -2f(x)\,dx = -2\int_0^2 f(x)\,dx = -2(-1.5) = 3.0$

55. $\displaystyle\int_0^6 [2 + f(x)]\,dx = \int_0^6 2\,dx + \int_0^6 f(x)\,dx$

$= 12 + 3.5 = 15.5$

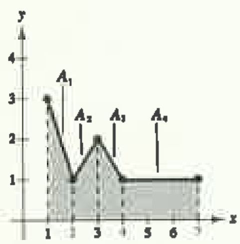
56. Average value $\displaystyle = \frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}(3.5) = 0.5833$

57. (a) $\displaystyle\int_1^7 f(x)\,dx =$ Sum of the areas

$= A_1 + A_2 + A_3 + A_4$

$= \frac{1}{2}(3 + 1) + \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 1) + (3)(1)$

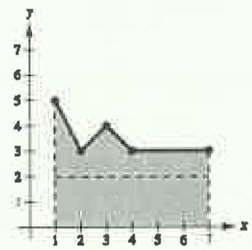
$= 8$



(b) Average value $\displaystyle = \frac{\int_1^7 f(x)\,dx}{7 - 1} = \frac{8}{6} = \frac{4}{3}$

(c) $A = 8 + (6)(2) = 20$

Average value $\displaystyle = \frac{20}{6} = \frac{10}{3}$



58. $P = 5\left(\sqrt{t} + 30\right)$

(a)

t	1	2	3	4	5	6
P	155	157.071	158.660	160	161.180	162.247

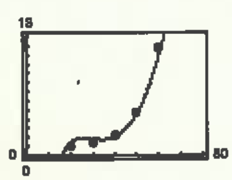
Average profit $\displaystyle \approx \frac{1}{6}(155 + 157.071 + 158.660 + 160 + 161.180 + 162.247) = \frac{954.158}{6} \approx 159.026$

(b) $\displaystyle \frac{1}{6}\int_{0.5}^{6.5} 5\left(\sqrt{t} + 30\right)dt = \frac{1}{6}\left[5\left(\frac{2}{3}t^{3/2} + 30t\right)\right]_{5.0}^{6.5} \approx \frac{954.061}{6} \approx 159.010$

(c) The definite integral yields a better approximation.

59. $R = -91.1 - 6.313x + 0.035x^2 + 45.794\sqrt{x},\ 20 \le x \le 60$

(a)



(b) $\displaystyle R'(x) = -6.313 + 0.07x + \frac{22.897}{\sqrt{x}}$

$R'(40) \approx 0.12$

$R'(50) \approx 0.43$

(c) $\displaystyle \frac{1}{10}\int_{30}^{40} R(x)\,dx \approx 1.80$ (between 30 and 40)

$\displaystyle \frac{1}{10}\int_{50}^{60} R(x)\,dx \approx 7.35$ (between 50 and 60)

60. $\displaystyle \frac{1}{R - 0}\int_0^R k(R^2 - r^2)\,dr = \frac{k}{R}\left[R^2r - \frac{r^3}{3}\right]_0^R = \frac{2kR^2}{3}$

61. (a) $F(x) = k\sec^2 x$

$F(0) = k = 500$

$F(x) = 500\sec^2 x$

(b) $\dfrac{1}{\pi/3 - 0}\displaystyle\int_0^{\pi/3} 500\sec^2 x\,dx = \frac{1500}{\pi}\Big[\tan x\Big]_0^{\pi/3}$

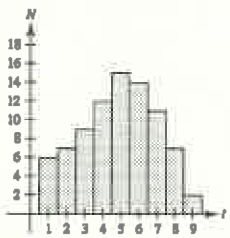
$= \dfrac{1500}{\pi}(\sqrt{3} - 0)$

≈ 826.99 newtons

≈ 827 newtons

62. $\dfrac{1}{5-0}\displaystyle\int_0^5 (0.1729t + 0.1552t^2 - 0.0374t^3)\,dt \approx \frac{1}{5}\Big[0.08645t^2 + 0.05073t^3 - 0.00935t^4\Big]_0^5 \approx 0.5318$ liter

63. (a) histogram

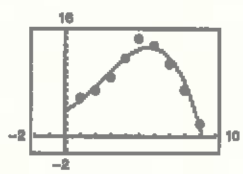


(b) $[6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980$ customers

(c) Using a graphing utility, you obtain

$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052t + 4.10317.$

(d)



(e) $\displaystyle\int_0^9 N(t)\,dt \approx 85.162$

The estimated number of customers is $(85.162)(60) \approx 5110$.

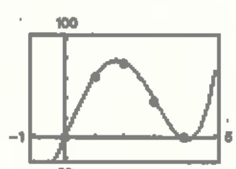
(f) Between 3 P.M. and 7 P.M., the number of customers is approximately

$\left(\displaystyle\int_3^7 N(t)\,dt\right)(60) \approx (50.28)(60) \approx 3017.$

Hence, $3017/240 \approx 12.6$ customers per minute.

64. (a) $R = 2.33t^4 - 14.67t^3 + 3.67t^2 + 70.67t$

(b)

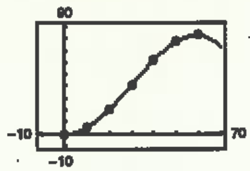


(c) $\displaystyle\int_0^4 R(t)\,dt = \left[\frac{2.33t^5}{5} - \frac{14.67t^4}{4} + \frac{3.67t^3}{3} + \frac{70.67t^2}{2}\right]_0^4$

$= 181.957$

65. (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$

(b)



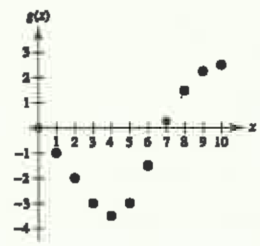
(c) $\displaystyle\int_0^{60} v(t)\,dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t\right]_0^{60} = 2472 \text{ meters}$

66. (a) $g(0) = \displaystyle\int_0^0 f(t)\,dt = 0.$

$g(1) = \displaystyle\int_0^1 f(t)\,dt = -1$ etc.

x	0	1	2	3	4	5	6	7	8	9	10
$g(x)$	0	−1	−2	−3	−3.5	−3	−1.5	0.25	1.5	2.25	2.5

(b)



(c) g has its minimum at $x = 4$.

(d) The first 4 points are collinear:

$(0, 0)$, $(1, -1)$, $(2, -2)$, and $(3, -3)$.

(e) Between $x = 5$ and $x = 6$.

67. (a) $\displaystyle\int_0^x (t+2)\,dt = \left[\frac{t^2}{2} + 2t\right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\dfrac{d}{dx}\left[\dfrac{1}{2}x^2 + 2x\right] = x + 2$

68. (a) $\displaystyle\int_0^x t(t^2+1)\,dt = \int_0^x (t^3 + t)\,dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2+2)$

(b) $\dfrac{d}{dx}\left[\dfrac{1}{4}x^4 + \dfrac{1}{2}x^2\right] = x^3 + x = x(x^2+1)$

69. (a) $\displaystyle\int_8^x \sqrt[3]{t}\,dt = \left[\frac{3}{4}t^{4/3}\right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\dfrac{d}{dx}\left[\dfrac{3}{4}x^{4/3} - 12\right] = x^{1/3} = \sqrt[3]{x}$

70. (a) $\displaystyle\int_4^x \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$

(b) $\dfrac{d}{dx}\left[\dfrac{2}{3}x^{3/2} - \dfrac{16}{3}\right] = x^{1/2} = \sqrt{x}$

71. (a) $\displaystyle\int_{\pi/4}^x \sec^2 t\,dt = \Big[\tan t\Big]_{\pi/4}^x = \tan x - 1$

(b) $\dfrac{d}{dx}[\tan x - 1] = \sec^2 x$

72. (a) $\displaystyle\int_{\pi/3}^x \sec t \tan t\,dt = \Big[\sec t\Big]_{\pi/3}^x = \sec x - 2$

(b) $\dfrac{d}{dx}[\sec x - 2] = \sec x \tan x$

73. $F(x) = \displaystyle\int_{-2}^x (t^2 - 2t)\,dt$

$F'(x) = x^2 - 2x$

74. $F(x) = \displaystyle\int_1^x \sqrt[4]{t}\,dt$

$F'(x) = \sqrt[4]{x}$

75. $F(x) = \displaystyle\int_{-1}^x \sqrt{t^4+1}\,dt$

$F'(x) = \sqrt{x^4+1}$

76. $F(x) = \int_0^x \tan^4 t\, dt$

$F'(x) = \tan^4 x$

77. $F(x) = \int_0^x t\cos t\, dt$

$F'(x) = x\cos x$

78. $F(x) = \int_1^x \frac{t^2}{t^2+1}\, dt$

$F'(x) = \frac{x^2}{x^2+1}$

79. $F(x) = \int_x^{x+2} (4t+1)\, dt$

$= \left[2t^2 + t\right]_x^{x+2}$

$= [2(x+2)^2 + (x+2)] - [2x^2 + x]$

$= 8x + 10$

$F'(x) = 8$

Alternate solution:

$F(x) = \int_x^{x+2} (4t+1)\, dt$

$= \int_x^0 (4t+1)\, dt + \int_0^{x+2} (4t+1)\, dt$

$= -\int_0^x (4t+1)\, dt + \int_0^{x+2} (4t+1)\, dt$

$F'(x) = -(4x+1) + 4(x+2) + 1 = 8$

80. $F(x) = \int_{-x}^x t^3\, dt = \left[\frac{t^4}{4}\right]_{-x}^x = 0$

$F'(x) = 0$

Alternate solution

$F(x) = \int_{-x}^x t^3\, dt$

$= \int_{-x}^0 t^3\, dt + \int_0^x t^3\, dt$

$= -\int_0^{-x} t^3\, dt + \int_0^x t^3\, dt$

$F'(x) = -(-x)^3(-1) + (x^3) = 0$

81. $F(x) = \int_0^{\sin x} \sqrt{t}\, dt = \left[\frac{2}{3}t^{3/2}\right]_0^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$

$F'(x) = (\sin x)^{1/2}\cos x = \cos x\sqrt{\sin x}$

Alternate solution

$F(x) = \int_0^{\sin x} \sqrt{t}\, dt$

$F'(x) = \sqrt{\sin x}\,\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$

82. $F(x) = \int_2^{x^2} \frac{1}{t^2}\, dt = \left[-\frac{1}{t}\right]_2^{x^2} = -\frac{1}{x^2} + \frac{1}{2}$

Alternate solution

$F(x) = \int_2^{x^2} \frac{1}{t^2}\, dt$

$F'(x) = \frac{1}{(x^2)^2}(2x) = \frac{2}{x^3}$

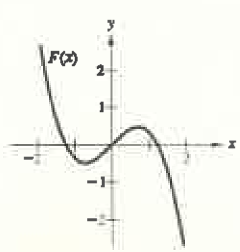
83. $F(x) = \int_0^{x^3} \sin t^2\, dt$

$F'(x) = \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6$

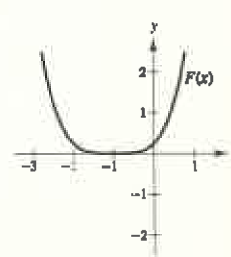
84. $F(x) = \int_0^{3x} \sqrt{1+t^3}\, dt$

$F'(x) = \sqrt{1+(3x)^3} \cdot 3 = 3\sqrt{1+27x^3}$

85. The extrema of F correspond to the zeros of f and the inflection point of F corresponds to the extrema of f.



86. The extremum of F corresponds to the zero of f. Since f has no extrema F has no points of inflection.



87. (a) $C(x) = 5000\left(25 + 3\int_0^x t^{1/4}\,dt\right)$

$= 5000\left(25 + 3\left[\frac{4}{5}t^{5/4}\right]_0^x\right)$

$= 5000\left(25 + \frac{12}{5}x^{5/4}\right) = 1000(125 + 12x^{5/4})$

(b) $C(1) = 1000(125 + 12(1)) = \$137{,}000$

$C(5) = 1000(125 + 12(5)^{5/4}) \approx \$214{,}721$

$C(10) = 1000(125 + 12(10)^{5/4}) \approx \$338{,}394$

88. (a) $g(t) = 4 - \frac{4}{t^2}$

$\lim_{t\to\infty} g(t) = 4$

Horizontal asymptote: $y = 4$

(b) $A(x) = \int_1^x \left(4 - \frac{4}{t^2}\right) dt$

$= \left[4t + \frac{4}{t}\right]_1^x = 4x + \frac{4}{x} - 8$

$= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$

$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\left(4x + \frac{4}{x} - 8\right) = \infty + 0 - 8 = \infty$

The graph of $A(x)$ does not have a horizontal asymptote.

89. True

90. True

91. False; $\int_{-1}^{1} x^{-2}\,dx = \int_{-1}^{0} x^{-2}\,dx + \int_0^1 x^{-2}\,dx$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

92. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t)\,dt = \Big[F(t)\Big]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = \frac{d}{dx}[F(v(x)) - F(u(x))]$$

$$= F'(v(x))v'(x) - F'(u(x))u'(x)$$

$$= f(v(x))v'(x) - f(u(x))u'(x).$$

93. $f(x) = \int_0^{1/x} \frac{1}{t^2+1}\,dt + \int_0^x \frac{1}{t^2+1}\,dt$

By the Second Fundamental Theorem of Calculus, we have

$$f'(x) = \frac{1}{(1/x)^2 + 1}\left(-\frac{1}{x^2}\right) + \frac{1}{x^2+1}$$

$$= -\frac{1}{1+x^2} + \frac{1}{x^2+1} = 0.$$

Since $f'(x) = 0$, $f(x)$ must be constant.

94. $G(x) = \int_0^x \left[s\int_0^s f(t)\,dt\right] ds$

(a) $G(0) = \int_0^0 \left[s\int_0^s f(t)\,dt\right] ds = 0$

(b) Let $F(s) = s\int_0^s f(t)\,dt$.

$G(x) = \int_0^x F(s)\,ds$

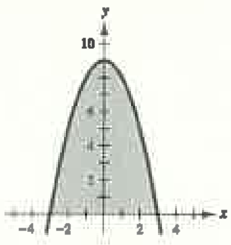
$G'(x) = F(x) = x\int_0^x f(t)\,dt$

$G'(0) = 0\int_0^0 f(t)\,dt = 0$

(c) $G''(x) = x \cdot f(x) + \int_0^x f(t)\,dt$

(d) $G''(0) = 0 \cdot f(0) + \int_0^0 f(t)\,dt = 0$

95. (a)

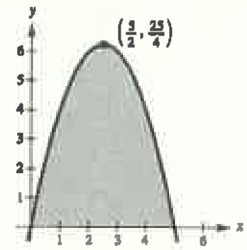


$$A = \int_{-3}^{3} (9 - x^2)\, dx = 2\int_0^3 (9 - x^2)\, dx$$

$$= 2\left[9x - \frac{x^3}{3}\right]_0^3 = 2\left[27 - \frac{27}{3}\right] = 36$$

(b) base = 6, height = 9, Area = $\frac{2}{3}(6)(9) = 36$

(c)



$$A = \int_0^5 (5x - x^2)\, dx = \left[\frac{5}{2}x^2 - \frac{x^3}{3}\right]_0^5 = \frac{125}{2} - \frac{125}{3} = \frac{125}{6}$$

Formula: $A = \frac{2}{3}(5)\left(\frac{25}{4}\right) = \frac{125}{6}$

96. $x(t) = t^3 - 6t^2 + 9t - 2$

$$x'(t) = 3t^2 - 12t + 9$$
$$= 3(t^2 - 4t + 3)$$
$$= 3(t - 3)(t - 1)$$

$$\text{Total distance} = \int_0^5 |x'(t)|\,dt$$
$$= \int_0^5 3|(t - 3)(t - 1)|\,dt$$
$$= 3\int_0^1 (t^2 - 4t + 3)dt - 3\int_1^3 (t^2 - 4t + 3)dt + 3\int_3^5 (t^2 - 4t + 3)dt$$
$$= 4 + 4 + 20$$
$$= 28 \text{ units}$$

97. $x(t) = (t - 1)(t - 3)^2 = t^3 - 7t^2 + 15t - 9$

$x'(t) = 3t^2 - 14t + 15$

Using a graphing utility,

$$\text{Total distance} = \int_0^5 |x'(t)|\,dt \approx 27.37 \text{ units}$$

98. $$\text{Total distance} = \int_1^4 |x'(t)|\,dt$$
$$= \int_1^4 |v(t)|\,dt$$
$$= \int_1^4 \frac{1}{\sqrt{t}}\,dt$$
$$= 2t^{1/2}\Big]_1^4$$
$$= 2(2 - 1) = 2 \text{ units}$$

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1. $\int (5x^2+1)^2(10x)\,dx$	$5x^2+1$	$10x\,dx$
2. $\int x^2\sqrt{x^3+1}\,dx$	x^3+1	$3x^2\,dx$
3. $\int \frac{x}{\sqrt{x^2+1}}\,dx$	x^2+1	$2x\,dx$
4. $\int \sec 2x \tan 2x\,dx$	$2x$	$2\,dx$
5. $\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$
6. $\int \frac{\cos x}{\sin^2 x}\,dx$	$\sin x$	$\cos x\,dx$

7. $\int (1+2x)^4\,2\,dx = \frac{(1+2x)^5}{5} + C$

Check: $\frac{d}{dx}\left[\frac{(1+2x)^5}{5} + C\right] = 2(1+2x)^4$

8. $\int (x^2-1)^3\,2x\,dx = \frac{(x^2-1)^4}{4} + C$

Check: $\frac{d}{dx}\left[\frac{(x^2-1)^4}{4} + C\right] = 2x(x^2-1)^3$

9. $\int (9-x^2)^{1/2}(-2x)\,dx = \frac{(9-x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9-x^2)^{3/2} + C$

Check: $\frac{d}{dx}\left[\frac{2}{3}(9-x^2)^{3/2} + C\right] = \frac{2}{3}\cdot\frac{3}{2}(9-x^2)^{1/2}(-2x) = \sqrt{9-x^2}(-2x)$

10. $\int (1-2x^2)^3(-4x)\,dx = \frac{(1-2x^2)^4}{4} + C$

Check: $\frac{d}{dx}\left[\frac{(1-2x^2)^4}{4} + C\right] = \frac{4(1-2x^2)^3(-4x)}{4} = (1-2x^2)^3(-4x)$

11. $\int x^2(x^3-1)^4\,dx = \frac{1}{3}\int (x^3-1)^4(3x^2)\,dx = \frac{1}{3}\left[\frac{(x^3-1)^5}{5}\right] + C = \frac{(x^3-1)^5}{15} + C$

Check: $\frac{d}{dx}\left[\frac{(x^3-1)^5}{15} + C\right] = \frac{5(x^3-1)^4(3x^2)}{15} = x^2(x^3-1)^4$

12. $\int x(4x^2+3)^3\,dx = \frac{1}{8}\int (4x^2+3)^3(8x)\,dx = \frac{1}{8}\left[\frac{(4x^2+3)^4}{4}\right] + C = \frac{(4x^2+3)^4}{32} + C$

Check: $\frac{d}{dx}\left[\frac{(4x^2+3)^4}{32} + C\right] = \frac{4(4x^2+3)^3(8x)}{32} = x(4x^2+3)^3$

13. $\int 5x(1-x^2)^{1/3}\,dx = -\frac{5}{2}\int (1-x^2)^{1/3}(-2x)\,dx = -\frac{5}{2}\cdot\frac{(1-x^2)^{4/3}}{4/3} + C = -\frac{15}{8}(1-x^2)^{4/3} + C$

Check: $\frac{d}{dx}\left[-\frac{15}{8}(1-x^2)^{4/3} + C\right] = -\frac{15}{8}\cdot\frac{4}{3}(1-x^2)^{1/3}(-2x) = 5x(1-x^2)^{1/3} = 5x\sqrt[3]{1-x^2}$

14. $\int u^3\sqrt{u^4+2}\,du = \frac{1}{4}\int (u^4+2)^{1/2}(4u^3)\,du = \frac{1}{4}\left[\frac{(u^4+2)^{3/2}}{3/2}\right] + C = \frac{1}{6}(u^4+2)^{3/2} + C$

Check: $\frac{d}{dx}\left[\frac{1}{6}(u^4+2)^{3/2} + C\right] = \frac{1}{6}\cdot\frac{3}{2}(u^4+2)^{1/2}(4u^3) = u^3\sqrt{u^4+2}$

15. $\int \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3}\int (1+x^3)^{-2}(3x^2)\,dx = \frac{1}{3}\left[\frac{(1+x^3)^{-1}}{-1}\right] + C = -\frac{1}{3(1+x^3)} + C$

Check: $\frac{d}{dx}\left[-\frac{1}{3(1+x^3)} + C\right] = -\frac{1}{3}(-1)(1+x^3)^{-2}(3x^2) = \frac{x^2}{(1+x^3)^2}$

16. $\int \frac{x^2}{(16-x^3)^2}\,dx = -\frac{1}{3}\int (16-x^3)^{-2}(-3x^2)\,dx = -\frac{1}{3}\left[\frac{(16-x^3)^{-1}}{-1}\right] + C = \frac{1}{3(16-x^3)} + C$

Check: $\frac{d}{dx}\left[\frac{1}{3(16-x^3)} + C\right] = \frac{1}{3}(-1)(16-x^3)^{-2}(3x^2) = \frac{x^2}{(16-x^3)^2}$

17. $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt = -\int\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right)dt = -\frac{[1+(1/t)]^4}{4} + C$

Check: $\frac{d}{dt}\left[-\frac{[1+(1/t)]^4}{4} + C\right] = -\frac{1}{4}(4)\left(1+\frac{1}{t}\right)^3\left(-\frac{1}{t^2}\right) = \frac{1}{t^2}\left(1+\frac{1}{t}\right)^3$

18. $\int\left[x^2 + \frac{1}{(3x)^2}\right]dx = \int\left(x^2 + \frac{1}{9}x^{-2}\right)dx = \frac{x^3}{3} + \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4-1}{9x} + C$

Check: $\frac{d}{dx}\left[\frac{1}{3}x^3 - \frac{1}{9}x^{-1} + C\right] = x^2 + \frac{1}{9}x^{-2} = x^2 + \frac{1}{(3x)^2}$

19. $\int \frac{1}{\sqrt{2x}}\,dx = \frac{1}{2}\int (2x)^{-1/2}\,2\,dx = \frac{1}{2}\left[\frac{(2x)^{1/2}}{1/2}\right] + C = \sqrt{2x} + C$

Check: $\frac{d}{dx}\left[\sqrt{2x} + C\right] = \frac{1}{2}(2x)^{-1/2}(2) = \frac{1}{\sqrt{2x}}$

20. $\int \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int x^{-1/2}\,dx = \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \sqrt{x} + C$

Check: $\frac{d}{dx}\left[\sqrt{x} + C\right] = \frac{1}{2\sqrt{x}}$

21. $\int \frac{x^2+3x+7}{\sqrt{x}}\,dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2})\,dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2+5x+35) + C$

Check: $\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C\right] = \frac{x^2+3x+7}{\sqrt{x}}$

22. $\int \frac{t+2t^2}{\sqrt{t}}\,dt = \int (t^{1/2} + 2t^{3/2})\,dt = \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C = \frac{2}{15}t^{3/2}(5+6t) + C$

Check: $\frac{d}{dt}\left[\frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C\right] = t^{1/2} + 2t^{3/2} = \frac{t+2t^2}{\sqrt{t}}$

23. $\displaystyle\int t^2\left(t - \frac{2}{t}\right) dt = \int (t^3 - 2t)\, dt = \frac{1}{4}t^4 - t^2 + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{1}{4}t^4 - t^2 + C\right] = t^3 - 2t = t^2\left(t - \frac{2}{t}\right)$

24. $\displaystyle\int\left(\frac{t^3}{3} + \frac{1}{4t^2}\right) dt = \int\left(\frac{1}{3}t^3 + \frac{1}{4}t^{-2}\right) dt = \frac{1}{3}\left(\frac{t^4}{4}\right) + \frac{1}{4}\left(\frac{t^{-1}}{-1}\right) + C = \frac{1}{12}t^4 - \frac{1}{4t} + C$

Check: $\displaystyle\frac{d}{dt}\left[\frac{1}{12}t^4 - \frac{1}{4t} + C\right] = \frac{1}{3}t^3 + \frac{1}{4t^2}$

25. $\displaystyle\int (9 - y)\sqrt{y}\, dy = \int (9y^{1/2} - y^{3/2})\, dy = 9\left(\frac{2}{3}y^{3/2}\right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$

Check: $\displaystyle\frac{d}{dy}\left[\frac{2}{5}y^{3/2}(15 - y) + C\right] = \frac{d}{dy}\left[6y^{3/2} - \frac{2}{5}y^{5/2} + C\right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$

26. $\displaystyle\int 2\pi y(8 - y^{3/2})\, dy = 2\pi\int (8y - y^{5/2})\, dy = 2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$

Check: $\displaystyle\frac{d}{dy}\left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C\right] = \frac{d}{dy}\left[2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C\right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$

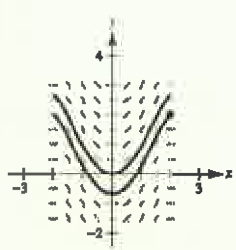
27.
$$\begin{aligned} y &= \int\left[4x + \frac{4x}{\sqrt{16 - x^2}}\right] dx \\ &= 4\int x\, dx - 2\int (16 - x^2)^{-1/2}(-2x)\, dx \\ &= 4\left(\frac{x^2}{2}\right) - 2\left[\frac{(16 - x^2)^{1/2}}{1/2}\right] + C \\ &= 2x^2 - 4\sqrt{16 - x^2} + C \end{aligned}$$

28.
$$\begin{aligned} y &= \int \frac{10x^2}{\sqrt{1 + x^3}}\, dx \\ &= \frac{10}{3}\int (1 + x^3)^{-1/2}(3x^2)\, dx \\ &= \frac{10}{3}\left[\frac{(1 + x^3)^{1/2}}{1/2}\right] + C \\ &= \frac{20}{3}\sqrt{1 + x^3} + C \end{aligned}$$

29.
$$\begin{aligned} y &= \int \frac{x + 1}{(x^2 + 2x - 3)^2}\, dx \\ &= \frac{1}{2}\int (x^2 + 2x - 3)^{-2}(2x + 2)\, dx \\ &= \frac{1}{2}\left[\frac{(x^2 + 2x - 3)^{-1}}{-1}\right] + C \\ &= -\frac{1}{2(x^2 + 2x - 3)} + C \end{aligned}$$

30.
$$\begin{aligned} y &= \int \frac{x - 4}{\sqrt{x^2 - 8x + 1}}\, dx \\ &= \frac{1}{2}\int (x^2 - 8x + 1)^{-1/2}(2x - 8)\, dx \\ &= \frac{1}{2}\left[\frac{(x^2 - 8x + 1)^{1/2}}{1/2}\right] + C \\ &= \sqrt{x^2 - 8x + 1} + C \end{aligned}$$

31. (a)



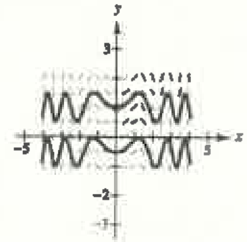
(b) $\displaystyle\frac{dy}{dx} = x\sqrt{4 - x^2},\ (2, 2)$

$$\begin{aligned} y &= \int x\sqrt{4 - x^2}\, dx = -\frac{1}{2}\int (4 - x^2)^{1/2}(-2x\, dx) \\ &= -\frac{1}{2}\cdot\frac{2}{3}(4 - x^2)^{3/2} + C = -\frac{1}{3}(4 - x^2)^{3/2} + C \end{aligned}$$

$(2, 2)$: $2 = -\frac{1}{3}(4 - 2^2)^{3/2} + C \Rightarrow C = 2$

$$y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$$

32. (a)



(b) $\frac{dy}{dx} = x\cos x^2, (0, 1)$

$$y = \int x\cos x^2\,dx = \frac{1}{2}\int \cos(x^2)2x\,dx$$

$$= \frac{1}{2}\sin(x^2) + C$$

$$(0, 1):\ 1 = \frac{1}{2}\sin(0) + C \Rightarrow C = 1$$

$$y = \frac{1}{2}\sin(x^2) + 1$$

33. $\int \sin 2x\,dx = \frac{1}{2}\int (\sin 2x)(2x)\,dx = -\frac{1}{2}\cos 2x + C$

34. $\int x\sin x^2\,dx = \frac{1}{2}\int (\sin x^2)(2x)\,dx = -\frac{1}{2}\cos x^2 + C$

35. $\int \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta = -\int \cos\frac{1}{\theta}\left(-\frac{1}{\theta^2}\right)d\theta = -\sin\frac{1}{\theta} + C$

36. $\int \cos 6x\,dx = \frac{1}{6}\int (\cos 6x)(6)\,dx = \frac{1}{6}\sin 6x + C$

37. $\int \sin 2x\cos 2x\,dx = \frac{1}{2}\int (\sin 2x)(2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^2}{2} + C = \frac{1}{4}\sin^2 2x + C$ OR

$\int \sin 2x\cos 2x\,dx = -\frac{1}{2}\int (\cos 2x)(-2\sin 2x)\,dx = -\frac{1}{2}\frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4}\cos^2 2x + C_1$ OR

$\int \sin 2x\cos 2x\,dx = \frac{1}{2}\int 2\sin 2x\cos 2x\,dx = \frac{1}{2}\int \sin 4x\,dx = \frac{1}{8}\cos 4x + C_2$

38. $\int \sec(1-x)\tan(1-x)\,dx = -\int [\sec(1-x)\tan(1-x)](-1)\,dx = -\sec(1-x) + C$

39. $\int \tan^4 x\sec^2 x\,dx = \frac{\tan^5 x}{5} + C = \frac{1}{5}\tan^5 x + C$

40. $\int \sqrt{\cot x}\csc^2 x\,dx = -\int (\cot x)^{1/2}(-\csc^2 x)\,dx = -\frac{2}{3}(\cot x)^{3/2} + C$

41. $\int \frac{\csc^2 x}{\cot^3 x}\,dx = -\int (\cot x)^{-3}(-\csc^2 x)\,dx$

$$= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2\cot^2 x} + C = \frac{1}{2}\tan^2 x + C = \frac{1}{2}(\sec^2 x - 1) + C = \frac{1}{2}\sec^2 x + C_1$$

42. $\int \frac{\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)dx = \int \sec x\tan x\,dx = \sec x + C$

43. $\int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -\cot x - x + C$

44. $\int \csc^2\left(\frac{x}{2}\right)dx = 2\int \csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = -2\cot\left(\frac{x}{2}\right) + C$

45. $f(x) = \int \cos\frac{x}{2}\,dx = 2\sin\frac{x}{2} + C$

Since $f(0) = 3 = 2\sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2\sin\frac{x}{2} + 3.$$

46. $f(x) = \int \pi\sec\pi x\tan\pi x\,dx = \sec\pi x + C$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$$f(x) = \sec\pi x - 1.$$

47. $u = x + 2, x = u - 2, dx = du$

$$\begin{aligned}\int x\sqrt{x+2}\,dx &= \int (u-2)\sqrt{u}\,du\\ &= \int (u^{3/2} - 2u^{1/2})\,du\\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C\\ &= \frac{2u^{3/2}}{15}(3u - 10) + C\\ &= \frac{2}{15}(x+2)^{3/2}[3(x+2) - 10] + C\\ &= \frac{2}{15}(x+2)^{3/2}(3x - 4) + C\end{aligned}$$

48. $u = 2x + 1, x = \frac{1}{2}(u - 1), dx = \frac{1}{2}\,du$

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\frac{1}{2}\,du\\ &= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C\\ &= \frac{u^{3/2}}{30}(3u - 5) + C\\ &= \frac{1}{30}(2x+1)^{3/2}[3(2x+1) - 5] + C\\ &= \frac{1}{30}(2x+1)^{3/2}(6x - 2) + C\\ &= \frac{1}{15}(2x+1)^{3/2}(3x - 1) + C\end{aligned}$$

49. $u = 1 - x, x = 1 - u, dx = -du$

$$\begin{aligned}\int x^2\sqrt{1-x}\,dx &= -\int (1-u)^2\sqrt{u}\,du\\ &= -\int (u^{1/2} - 2u^{3/2} + u^{5/2})\,du\\ &= -\left(\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right) + C\\ &= -\frac{2u^{3/2}}{105}(35 - 42u + 15u^2) + C\\ &= -\frac{2}{105}(1-x)^{3/2}[35 - 42(1-x) + 15(1-x)^2] + C\\ &= -\frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C\end{aligned}$$

50. $u = 2 - x, x = 2 - u, dx = -du$

$$\begin{aligned}\int (x+1)\sqrt{2-x}\,dx &= -\int (3-u)\sqrt{u}\,du\\ &= -\int (3u^{1/2} - u^{3/2})\,du\\ &= -\left(2u^{3/2} - \frac{2}{5}u^{5/2}\right) + C\\ &= -\frac{2u^{3/2}}{5}(5 - u) + C\\ &= -\frac{2}{5}(2-x)^{3/2}[5 - (2-x)] + C\\ &= -\frac{2}{5}(2-x)^{3/2}(x + 3) + C\end{aligned}$$

51. $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2}\,du$

$$\begin{aligned}\int \frac{x^2 - 1}{\sqrt{2x - 1}}\,dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}}\frac{1}{2}\,du\\ &= \frac{1}{8}\int u^{-1/2}[(u^2 + 2u + 1) - 4]\,du\\ &= \frac{1}{8}\int (u^{3/2} + 2u^{1/2} - 3u^{-1/2})\,du\\ &= \frac{1}{8}\left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} - 6u^{1/2}\right) + C\\ &= \frac{u^{1/2}}{60}(3u^2 + 10u - 45) + C\\ &= \frac{\sqrt{2x - 1}}{60}[3(2x - 1)^2 + 10(2x - 1) - 45] + C\\ &= \frac{1}{60}\sqrt{2x - 1}(12x^2 + 8x - 52) + C\\ &= \frac{1}{15}\sqrt{2x - 1}(3x^2 + 2x - 13) + C\end{aligned}$$

52. $u = x + 3, x = u - 3, dx = du$

$$\begin{aligned}\int \frac{2x - 1}{\sqrt{x + 3}}\,dx &= \int \frac{2(u - 3) - 1}{\sqrt{u}}\,du\\ &= \int (2u^{1/2} - 7u^{-1/2})\,du\\ &= \frac{4}{3}u^{3/2} - 14u^{1/2} + C\\ &= \frac{2u^{1/2}}{3}(2u - 21) + C\\ &= \frac{2}{3}\sqrt{x + 3}[2(x + 3) - 21] + C\\ &= \frac{2}{3}\sqrt{x + 3}(2x - 15) + C\end{aligned}$$

53. $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned}\int \frac{-x}{(x + 1) - \sqrt{x + 1}}\,dx &= \int \frac{-(u - 1)}{u - \sqrt{u}}\,du\\ &= -\int \frac{(\sqrt{u} + 1)(\sqrt{u} - 1)}{\sqrt{u}(\sqrt{u} - 1)}\,du\\ &= -\int (1 + u^{-1/2})\,du\\ &= -(u + 2u^{1/2}) + C\\ &= -u - 2\sqrt{u} + C\\ &= -(x + 1) - 2\sqrt{x + 1} + C\\ &= -x - 2\sqrt{x + 1} - 1 + C\\ &= -(x + 2\sqrt{x + 1}) + C_1\end{aligned}$$

where $C_1 = -1 + C$.

54. $u = t - 4, t = u + 4, dt = du$

$$\begin{aligned}\int t\sqrt[3]{t - 4}\,dt &= \int (u + 4)u^{1/3}\,du\\ &= \int (u^{4/3} + 4u^{1/3})\,du\\ &= \frac{3}{7}u^{7/3} + 3u^{4/3} + C\\ &= \frac{3u^{4/3}}{7}(u + 7) + C\\ &= \frac{3}{7}(t - 4)^{4/3}[(t - 4) + 7] + C\\ &= \frac{3}{7}(t - 4)^{4/3}(t + 3) + C\end{aligned}$$

55. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int_{-1}^{1} x(x^2+1)^3\,dx = \frac{1}{2}\int_{-1}^{1}(x^2+1)^3(2x)\,dx = \left[\frac{1}{8}(x^2+1)^4\right]_{-1}^{1} = 0$$

56. Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_{0}^{1} x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_{0}^{1}(1-x^2)^{1/2}(-2x)\,dx = \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_{0}^{1} = 0 + \frac{1}{3} = \frac{1}{3}$$

57. Let $u = 2x + 1$, $du = 2\,dx$.

$$\int_{0}^{4} \frac{1}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int_{0}^{4}(2x+1)^{-1/2}(2)\,dx = \left[\sqrt{2x+1}\right]_{0}^{4} = \sqrt{9} - \sqrt{1} = 2$$

58. Let $u = 1 + 2x^2$, $du = 4x\,dx$.

$$\int_{0}^{2} \frac{x}{\sqrt{1+2x^2}}\,dx = \frac{1}{4}\int_{0}^{2}(1+2x^2)^{-1/2}(4x)\,dx = \left[\frac{1}{2}\sqrt{1+2x^2}\right]_{0}^{2} = \frac{3}{2} - \frac{1}{2} = 1$$

59. Let $u = 1 + \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}\,dx$.

$$\int_{1}^{9} \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int_{1}^{9}(1+\sqrt{x})^{-2}\left(\frac{1}{2\sqrt{x}}\right)dx = \left[-\frac{2}{1+\sqrt{x}}\right]_{1}^{9} = -\frac{1}{2} + 1 = \frac{1}{2}$$

60. Let $u = 4 + x^2$, $du = 2x\,dx$.

$$\int_{0}^{2} x\sqrt[3]{4+x^2}\,dx = \frac{1}{2}\int_{0}^{2}(4+x^2)^{1/3}(2x)\,dx = \left[\frac{3}{8}(4+x^2)^{4/3}\right]_{0}^{2} = \frac{3}{8}(8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2}\sqrt[3]{4} \approx 3.619$$

61. $u = 2 - x$, $x = 2 - u$, $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_{1}^{2}(x-1)\sqrt{2-x}\,dx = \int_{1}^{0} -[(2-u)-1]\sqrt{u}\,du = \int_{1}^{0}(u^{3/2} - u^{1/2})\,du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1}^{0} = -\left[\frac{2}{5} - \frac{2}{3}\right] = \frac{4}{15}$$

62. $u = 2x + 1$, $x = \dfrac{1}{2}(u-1)$, $dx = \dfrac{1}{2}\,du$

When $x = 0$, $u = 1$. When $x = 4$, $u = 9$.

$$\int_{0}^{4}\frac{x}{\sqrt{2x+1}}\,dx = \int_{1}^{9}\frac{(1/2)(u-1)}{\sqrt{u}}\,\frac{1}{2}\,du = \frac{1}{4}\int_{1}^{9}(u^{1/2} - u^{-1/2})\,du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_{1}^{9} = \frac{1}{4}(18-6) - \frac{1}{4}\left(\frac{2}{3} - 2\right) = \frac{10}{3}$$

63. $$\int_{0}^{\pi/2}\cos\left(\frac{2}{3}x\right)dx = \left[\frac{3}{2}\sin\left(\frac{2}{3}x\right)\right]_{0}^{\pi/2} = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}$$

64. $$\int_{\pi/3}^{\pi/2}(x + \cos x)\,dx = \left[\frac{x^2}{2} + \sin x\right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1\right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2}\right) = \frac{5\pi^2}{72} + \frac{2-\sqrt{3}}{2}$$

65. $u = x + 1, x = u - 1, dx = du$

When $x = 0, u = 1$. When $x = 7, u = 8$.

$$\text{Area} = \int_0^7 x\sqrt[3]{x+1}\,dx = \int_1^8 (u-1)\sqrt[3]{u}\,du$$

$$= \int_1^8 (u^{4/3} - u^{1/3})\,du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3}\right]_1^8 = \left(\frac{384}{7} - 12\right) - \left(\frac{3}{7} - \frac{3}{4}\right) = \frac{1209}{28}$$

66. $u = x + 2, x = u - 2, dx = du$

When $x = -2, u = 0$. When $x = 6, u = 8$.

$$\text{Area} = \int_{-2}^6 x^2\sqrt[3]{x+2}\,dx = \int_0^8 (u-2)^2\sqrt[3]{u}\,du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3})\,du = \left[\frac{3}{10}u^{10/3} - \frac{12}{7}u^{7/3} + 3u^{4/3}\right]_0^8 = \frac{4752}{35}$$

67. $A = \displaystyle\int_0^{\pi} (2\sin x + \sin 2x)\,dx = -\left[2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi} = 4$

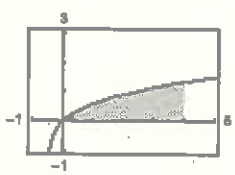
68. $A = \displaystyle\int_0^{\pi} (\sin x + \cos 2x)\,dx = \left[-\cos x + \frac{1}{2}\sin 2x\right]_0^{\pi} = 2$

69. $\text{Area} = \displaystyle\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)dx = 2\int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = \left[2\tan\left(\frac{x}{2}\right)\right]_{\pi/2}^{2\pi/3} = 2(\sqrt{3} - 1)$

70. Let $u = 2x, du = 2\,dx$.

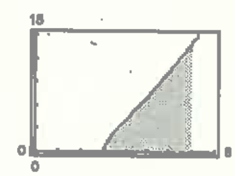
$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx = \frac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x(2)\,dx = \left[-\frac{1}{2}\csc 2x\right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

71. $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx \approx 3.333 = \frac{10}{3}$

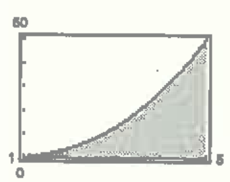


72. $\displaystyle\int_0^2 x^3\sqrt{x+2}\,dx \approx 7.581$

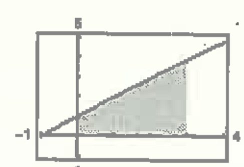
73. $\displaystyle\int_3^7 x\sqrt{x-3}\,dx \approx 28.8$



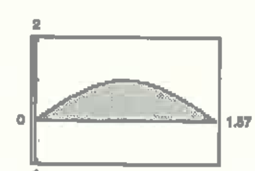
74. $\displaystyle\int_1^5 x^2\sqrt{x-1}\,dx \approx 67.505$



75. $\displaystyle\int_0^3 \left(\theta + \cos\frac{\theta}{6}\right)d\theta \approx 7.377$



76. $\displaystyle\int_0^{\pi/2} \sin 2x\,dx \approx 1.0$



77. $\displaystyle\int (2x-1)^2\,dx = \frac{1}{2}\int (2x-1)^2\,2\,dx = \frac{1}{6}(2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1$

$\displaystyle\int (2x-1)^2\,dx = \int (4x^2 - 4x + 1)\,dx = \frac{4}{3}x^3 - 2x^2 + x + C_2$

They differ by a constant: $C_2 = C_1 - \dfrac{1}{6}$.

78. $\displaystyle\int \sin x \cos x\, dx = \int (\sin x)^1(\cos x\, dx) = \frac{\sin^2 x}{2} + C_1$

$\displaystyle\int \sin x \cos x\, dx = -\int (\cos x)^1(-\sin x\, dx) = -\frac{\cos^2 x}{2} + C_2$

$\displaystyle -\frac{\cos^2 x}{2} + C_2 = -\frac{(1 - \sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

79. $\displaystyle\int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$; the function x^2 is an even function.

(a) $\displaystyle\int_{-2}^0 x^2\, dx = \int_0^2 x^2\, dx = \frac{8}{3}$

(b) $\displaystyle\int_{-2}^2 x^2\, dx = 2\int_0^2 x^2\, dx = \frac{16}{3}$

(c) $\displaystyle\int_0^2 (-x^2)\, dx = -\int_0^2 x^2\, dx = -\frac{8}{3}$

(d) $\displaystyle\int_{-2}^0 3x^2\, dx = 3\int_0^2 x^2\, dx = 8$

80. (a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin x\, dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx = 2\int_0^{\pi/4} \cos x\, dx = \Big[2 \sin x\Big]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the y-axis.

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\, dx = 2\int_0^{\pi/2} \cos x\, dx = \Big[2 \sin x\Big]_0^{\pi/2} = 2$

(d) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx = 0$ since $\sin(-x)\cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

81. $\displaystyle\int_{-4}^4 (x^3 + 6x^2 - 2x - 3)\, dx = \int_{-4}^4 (x^3 - 2x)\, dx + \int_{-4}^4 (6x^2 - 3)\, dx = 0 + 2\int_0^4 (6x^2 - 3)\, dx = 2\Big[2x^3 - 3x\Big]_0^4 = 232$

82. $\displaystyle\int_{-\pi}^{\pi} (\sin 3x + \cos 3x)\, dx = \int_{-\pi}^{\pi} \sin 3x\, dx + \int_{-\pi}^{\pi} \cos 3x\, dx = 0 + 2\int_0^{\pi} \cos 3x\, dx = \left[\frac{2}{3} \sin 3x\right]_0^{\pi} = 0$

83. $\displaystyle\frac{dV}{dt} = \frac{k}{(t+1)^2}$

$\displaystyle V(t) = \int \frac{k}{(t+1)^2}\, dt = -\frac{k}{t+1} + C$

$V(0) = -k + C = 500{,}000$

$V(1) = -\frac{1}{2}k + C = 400{,}000$

Solving this system yields $k = -200{,}000$ and $C = 300{,}000$. Thus,

$\displaystyle V(t) = \frac{200{,}000}{t+1} + 300{,}000.$

When $t = 4$, $V(4) = \$340{,}000$.

84. $\displaystyle\frac{dQ}{dt} = k(100 - t)^2$

$\displaystyle Q(t) = \int k(100 - t)^2\, dt = -\frac{k}{3}(100 - t)^3 + C$

$Q(100) = C = 0$

$Q(t) = -\frac{k}{3}(100 - t)^3$

$Q(0) = -\frac{k}{3}(100)^3 = 2{,}000{,}000 \Rightarrow k = -6$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) = \$250{,}000$.

85. $\dfrac{dC}{dx} = \dfrac{12}{\sqrt[3]{12x+1}}$

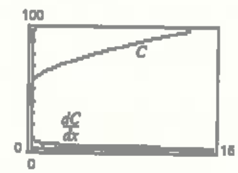
(a) $C(x) = \displaystyle\int \frac{12}{\sqrt[3]{12x+1}}\,dx$

$= \displaystyle\int (12x+1)^{-1/3}\,12\,dx = \frac{3}{2}(12x+1)^{2/3} + C_1$

$C(13) \approx 43.65 + C_1 = 100$

$C_1 = 56.35$

$C(x) = \dfrac{3}{2}(12x+1)^{2/3} + 56.35$

(b) 

86. $\displaystyle\frac{1}{b-a}\int_a^b \left[217 + 13\cos\frac{\pi(t-3)}{6}\right] dt = \frac{1}{b-a}\left[217t + \frac{78}{\pi}\sin\left(\frac{\pi(t-3)}{6}\right)\right]_a^b$

(a) $\displaystyle\frac{1}{3}\left[217t + \frac{78}{\pi}\sin\left(\frac{\pi(t-3)}{6}\right)\right]_0^3 = \frac{1}{3}\left(651 + \frac{78}{\pi}\right) \approx 225.28$ million barrels

(b) $\displaystyle\frac{1}{3}\left[217t + \frac{78}{\pi}\sin\left(\frac{\pi(t-3)}{6}\right)\right]_3^6 = \frac{1}{3}\left(1302 + \frac{78}{\pi} - 651\right) \approx 225.28$ million barrels

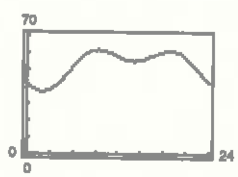
(c) $\displaystyle\frac{1}{12}\left[217t + \frac{78}{\pi}\sin\left(\frac{\pi(t-3)}{6}\right)\right]_0^{12} = \frac{1}{12}\left(2604 - \frac{78}{\pi} + \frac{78}{\pi}\right) = 217$ million barrels

87. $\displaystyle\frac{1}{b-a}\int_a^b \left[74.50 + 43.75\sin\frac{\pi t}{6}\right] dt = \frac{1}{b-a}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_a^b$

(a) $\displaystyle\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^3 = \frac{1}{3}\left(223.5 + \frac{262.5}{\pi}\right) \approx 102.352$ thousand units

(b) $\displaystyle\frac{1}{3}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_3^6 = \frac{1}{3}\left(447 + \frac{262.5}{\pi} - 223.5\right) \approx 102.352$ thousand units

(c) $\displaystyle\frac{1}{12}\left[74.50t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right]_0^{12} = \frac{1}{12}\left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi}\right) = 74.5$ thousand units

88. (a) 

Maximum flow: $R \approx 61.713$ at $t = 9.36$.
[(18.861, 61.178) is a relative maximum.]

(b) Volume $= \displaystyle\int_0^{24} R(t)\,dt \approx 1272$ (5 thousand of gallons)

89. $\displaystyle\frac{1}{b-a}\int_a^b [2\sin(60\pi t) + \cos(120\pi t)]\,dt = \frac{1}{b-a}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\,\pi}\sin(120\pi t)\right]_a^b$

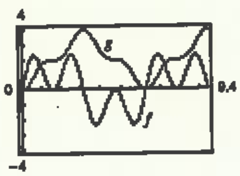
(a) $\displaystyle\frac{1}{(1/60)-0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/60} = 60\left[\left(\frac{1}{30\pi} + 0\right) - \left(-\frac{1}{30\pi}\right)\right] = \frac{4}{\pi} \approx 1.273$ amps

(b) $\displaystyle\frac{1}{(1/240)-0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/240} = 240\left[\left(-\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi}\right) - \left(-\frac{1}{30\pi}\right)\right]$

$\displaystyle = \frac{2}{\pi}(5 - 2\sqrt{2}) \approx 1.382$ amps

(c) $\displaystyle\frac{1}{(1/30)-0}\left[-\frac{1}{30\pi}\cos(60\pi t) + \frac{1}{120\pi}\sin(120\pi t)\right]_0^{1/30} = 30\left[\left(\frac{1}{30\pi}\right) - \left(-\frac{1}{30\pi}\right)\right] = 0$ amps

90. (a)

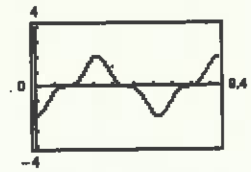


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g.

(d) No, some zeros of f, like $x = \pi/2$, do not correspond to an extrema of g. The graph of g continues to increase after $x = \pi/2$ because f remains above the x-axis.

(e)



The graph of h is that of g shifted 2 units downward.

$$\begin{aligned} g(t) &= \int_0^t f(x)\,dx \\ &= \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^t f(x)\,dx = 2 + h(t). \end{aligned}$$

91. False

$$\int (2x+1)^2\,dx = \frac{1}{2}\int (2x+1)^2\,2\,dx = \frac{1}{6}(2x+1)^3 + C$$

92. False

$$\int x(x^2+1)^2\,dx = \frac{1}{2}\int (x^2+1)(2x)\,dx = \frac{1}{4}(x^2+1)^2 + C$$

93. True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d)\,dx = \underbrace{\int_{-10}^{10} (ax^3 + cx)\,dx}_{\text{Odd}} + \underbrace{\int_{-10}^{10} (bx^2 + d)\,dx}_{\text{Even}} = 0 + 2\int_0^{10} (bx^2 + d)\,dx$$

94. True

$$\int_a^b \sin x\,dx = \Big[-\cos x\Big]_a^b = -\cos b + \cos a = -\cos(b + 2\pi) + \cos a = \int_a^{b+2\pi} \sin x\,dx$$

95. True

$$4\int \sin x \cos x\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

96. False

$$\int \sin^2 2x \cos 2x\,dx = \frac{1}{2}\int (\sin 2x)^2(2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^3}{3} + C = \frac{1}{6}\sin^3 2x + C$$

97. Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x+h)\,dx = \int_{a+h}^{b+h} f(u)\,du = \int_{a+h}^{b+h} f(x)\,dx.$$

98. $\displaystyle\int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx$

Let $u = b - x$, then $du = -dx$. When $x = 0$, $u = b$. When $x = b$, $u = 0$.

$$\int_0^b \frac{f(x)}{f(x) + f(b-x)}\,dx = -\int_b^0 \frac{f(b-u)}{f(b-u) + f(u)}\,du = \int_0^b \frac{f(b-x)}{f(b-x) + f(x)}\,dx$$

—CONTINUED—

98. —CONTINUED—

Also, by division,

$$\int_0^b \frac{f(x)}{f(x)+f(b-x)}\,dx = \int_0^b \left[1 - \frac{f(b-x)}{f(x)+f(b-x)}\right]dx.$$

Thus,

$$\int_0^b \frac{f(x)}{f(x)+f(b-x)}\,dx = \int_0^b 1\,dx - \int_0^b \frac{f(b-x)}{f(x)+f(b-x)}\,dx$$

$$\int_0^b \frac{f(x)}{f(x)+f(b-x)}\,dx + \int_0^b \frac{f(b-x)}{f(x)+f(b-x)}\,dx = \int_0^b 1\,dx$$

$$2\int_0^b \frac{f(x)}{f(x)+f(b-x)}\,dx = \Big[x\Big]_0^b = b.$$

Therefore,

$$\int_0^b \frac{f(x)}{f(x)+f(b-x)}\,dx = \frac{b}{2}$$

$$\int_0^1 \frac{\sin x}{\sin x + \sin(1-x)}\,dx = \frac{1}{2} \text{ (Let } f(x) = \sin x \text{ and } b = 1).$$

Section 4.6 Numerical Integration

1. Exact: $\displaystyle\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Simpson's: $\displaystyle\int_0^2 x^2\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^2 + 2(1)^2 + 4\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{8}{3} \approx 2.6667$

2. Exact: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx = \left[\frac{x^3}{6}+x\right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx \approx \frac{1}{8}\left[1 + 2\left(\frac{(1/4)^2}{2}+1\right) + 2\left(\frac{(1/2)^2}{2}+1\right) + 2\left(\frac{(3/4)^2}{2}+1\right) + \left(\frac{1^2}{2}+1\right)\right] = \frac{75}{64} \approx 1.1719$

Simpson's: $\displaystyle\int_0^1 \left(\frac{x^2}{2}+1\right)dx \approx \frac{1}{12}\left[1 + 4\left(\frac{(1/4)^2}{2}+1\right) + 2\left(\frac{(1/2)^2}{2}+1\right) + 4\left(\frac{(3/4)^2}{2}+1\right) + \left(\frac{1^2}{2}+1\right)\right] = \frac{7}{6} \approx 1.1667$

3. Exact: $\displaystyle\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.000$

Trapezoidal: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Simpson's: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 2(1)^3 + 4\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{24}{6} = 4.0000$

4. Exact: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^2 = 0.5000$

Trapezoidal: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{8}\left[1 + 2\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 2\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5090$

Simpson's: $\displaystyle\int_1^2 \frac{1}{x^2}\,dx \approx \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 4\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5004$

5. Exact: $\displaystyle\int_0^2 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^2 = 4.0000$

Trapezoidal: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{8}\left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8\right] = 4.0625$

Simpson's: $\displaystyle\int_0^2 x^3\,dx \approx \frac{1}{12}\left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8\right] = 4.0000$

6. Exact: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\displaystyle\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{3}[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

7. Exact: $\displaystyle\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\displaystyle\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right]$

≈ 12.6640

Simpson's: $\displaystyle\int_4^9 \sqrt{x}\,dx \approx \frac{5}{24}\left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3\right] \approx 12.6667$

8. Exact: $\displaystyle\int_1^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\displaystyle\int_1^3 (4 - x^2)\,dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] - 5\right\} = -0.7500$

Simpson's: $\displaystyle\int_1^3 (4 - x^2)\,dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 5\right] \approx -0.6667$

9. Exact: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx = \left[-\frac{1}{x+1}\right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{8}\left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$\displaystyle = \frac{1}{8}\left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9}\right) \approx 0.1676$

Simpson's: $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx \approx \frac{1}{12}\left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9}\right]$

$\displaystyle = \frac{1}{12}\left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9}\right) \approx 0.1667$

10. Exact: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx = \frac{1}{3}\left[(x^2+1)^{3/2}\right]_0^2 = \frac{1}{3}(5^{3/2} - 1) \approx 3.393$

Trapezoidal: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 2\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.457$

Simpson's: $\displaystyle\int_0^2 x\sqrt{x^2+1}\,dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{(1/2)^2+1} + 2(1)\sqrt{1^2+1} + 4\left(\frac{3}{2}\right)\sqrt{(3/2)^2+1} + 2\sqrt{2^2+1}\right] \approx 3.392$

11. Trapezoidal: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{4}[1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3}\,dx \approx \frac{1}{6}[1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 2\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1+(1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1+1^3}}\right) + 4\left(\frac{1}{\sqrt{1+(3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

13. $\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \int_0^1 \sqrt{x(1-x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)}\,dx \approx \frac{1}{12}\left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)}\right] \approx 0.372$

Graphing utility: 0.393

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

15. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{8}\left[\cos 0 + 2\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

≈ 0.957

Simpson's: $\int_0^{\sqrt{\pi/2}} \cos(x^2)\,dx \approx \frac{\sqrt{\pi/2}}{12}\left[\cos 0 + 4\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4\cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2\right]$

≈ 0.978

Graphing utility: 0.977

16. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.271

Simpson's: $\int_0^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.257

Graphing utility: 0.256

17. Trapezoidal: $\displaystyle\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{80}[\sin(1) + 2\sin(1.025)^2 + 2\sin(1.05)^2 + 2\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Simpson's: $\displaystyle\int_1^{1.1} \sin x^2\,dx \approx \frac{1}{120}[\sin(1) + 4\sin(1.025)^2 + 2\sin(1.05)^2 + 4\sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Graphing utility: 0.089

18. Trapezoidal: $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{16}\left[\sqrt{2} + 2\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 2\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Simpson's: $\displaystyle\int_0^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{24}\left[\sqrt{2} + 4\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 4\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Graphing utility: 1.910

19. Trapezoidal: $\displaystyle\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{32}\left[0 + 2\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 2\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.194$

Simpson's: $\displaystyle\int_0^{\pi/4} x\tan x\,dx \approx \frac{\pi}{48}\left[0 + 4\left(\frac{\pi}{16}\right)\tan\left(\frac{\pi}{16}\right) + 2\left(\frac{2\pi}{16}\right)\tan\left(\frac{2\pi}{16}\right) + 4\left(\frac{3\pi}{16}\right)\tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4}\right] \approx 0.186$

Graphing utility: 0.186

20. Trapezoidal: $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

21. $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\le \dfrac{(2-0)^3}{12(4^2)}(12) = 0.5$ since $f''(x)$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\le \dfrac{(2-0)^5}{180(4^4)}(0) = 0$ since $f^{(4)}(x) = 0$.

22. $f(x) = \dfrac{1}{x+1}$

$f'(x) = \dfrac{-1}{(x+1)^2}$

$f''(x) = \dfrac{2}{(x+1)^3}$

$f'''(x) = \dfrac{-6}{(x+1)^4}$

$f^{(4)}(x) = \dfrac{24}{(x+1)^5}$

(a) Trapezoidal: Error $\le \dfrac{(1-0)^2}{12(4^2)}(2) = \dfrac{1}{96} \approx 0.01$ since $f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\le \dfrac{(1-0)^5}{180(4^4)}(24) = \dfrac{1}{1920} \approx 0.0005$ since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

23. $f''(x) = \dfrac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\leq \dfrac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133{,}333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \dfrac{24}{x^5}$ in $[1, 3]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and when $|f^{(4)}(1)| = 24$.

Simpson's: Error $\leq \dfrac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426{,}666.67$, $n > 25.56$; let $n = 26$.

24. $f''(x) = \dfrac{2}{(1 + x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\leq \dfrac{1}{12n^2}(2) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

$f^{(4)}(x) = \dfrac{24}{(1 + x)^5}$ in $[0, 1]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\leq \dfrac{1}{180n^4}(24) < 0.00001$, $n^4 > 13{,}333.33$, $n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

25. $f(x) = \sqrt{1 + x}$

(a) $f''(x) = -\dfrac{1}{4(1 + x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{1}{4}$.

Trapezoidal: Error $\leq \dfrac{8}{12n^2}\left(\dfrac{1}{4}\right) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \dfrac{-15}{16(1 + x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{15}{16}$.

Simpson's: Error $\leq \dfrac{32}{180n^4}\left(\dfrac{15}{16}\right) < 0.00001$, $n^4 > 16{,}666.67$, $n > 11.36$; let $n = 12$.

26. $f(x) = (x + 1)^{2/3}$

(a) $f''(x) = -\dfrac{2}{9(x + 1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \dfrac{2}{9}$.

Trapezoidal: Error $\leq \dfrac{8}{12n^4}\left(\dfrac{2}{9}\right) < 0.00001$, $n^2 > 14{,}814.81$, $n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\dfrac{56}{81(x + 1)^{10/3}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \dfrac{56}{81}$.

Simpson's: Error $\leq \dfrac{32}{180n^4}\left(\dfrac{56}{81}\right) < 0.00001$, $n^4 > 12{,}290.81$, $n > 10.53$; let $n = 12$. (In Simpson's Rule n must be

27. $f(x) = \tan(x^2)$

(a) $f''(x) = 2\sec^2(x^2)[1 + 4x^2\tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\le \dfrac{(1-0)^3}{12n^2}(49.5305) < 0.00001$, $n^2 > 412{,}754.17$, $n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8\sec^2(x^2)[12x^2 + (3 + 32x^4)\tan(x^2) + 36x^2\tan^2(x^2) + 48x^4\tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\le \dfrac{(1-0)^5}{180n^4}(9184.4734) < 0.00001$, $n^4 > 5{,}102{,}485.22$, $n > 47.53$; let $n = 48$.

28. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2\sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\le \dfrac{(1-0)^3}{12n^2}(2.2853) < 0.00001$, $n^2 > 19{,}044.17$, $n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\le \dfrac{(1-0)^5}{180n^4}(28.4285) < 0.00001$, $n^4 > 15{,}793.61$, $n > 11.21$; let $n = 12$.

29. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\le \dfrac{(b-a)^5}{180n^4}(0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\displaystyle\int_0^1 x^3\,dx = \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)^3 + 1\right] = \frac{1}{4}$

This is the exact value of the integral.

30. The program will vary depending upon the computer or programmable calculator that you use.

31. $f(x) = \sqrt{2 + 3x^2}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

32. $f(x) = \sqrt{1 - x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

33. $f(x) = \sin\sqrt{x}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

34. $f(x) = \dfrac{\sin x}{x}$ on $[1, 2]$.

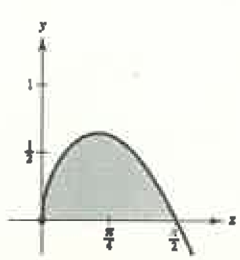
n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.7070	0.6597	0.6103	0.6586	0.6593
8	0.6833	0.6594	0.6350	0.6592	0.6593
10	0.6786	0.6594	0.6399	0.6592	0.6593
12	0.6754	0.6594	0.6431	0.6593	0.6593
16	0.6714	0.6594	0.6472	0.6593	0.6593
20	0.6690	0.6593	0.6496	0.6593	0.6593

35. $A = \displaystyle\int_0^{\pi/2} \sqrt{x}\cos x\,dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x}\cos x\,dx \approx \frac{\pi}{84}\left[\sqrt{0}\cos 0 + 4\sqrt{\frac{\pi}{28}}\cos\frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}}\cos\frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}}\cos\frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}}\cos\frac{\pi}{2}\right]$$

$$\approx 0.701$$



36. Simpson's Rule: $n = 8$

$$8\sqrt{3}\int_0^{\pi/2}\sqrt{1-\frac{2}{3}\sin^2\theta}\,d\theta \approx \frac{\sqrt{3}\pi}{6}\left[\sqrt{1-\frac{2}{3}\sin^2 0}+4\sqrt{1-\frac{2}{3}\sin^2\frac{\pi}{16}}+2\sqrt{1-\frac{2}{3}\sin^2\frac{\pi}{8}}+\cdots+\sqrt{1-\frac{2}{3}\sin^2\frac{\pi}{2}}\right]$$
$$\approx 17.476$$

37. $W = \int_0^5 100x\sqrt{125-x^3}\,dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x\sqrt{125-x^3}\,dx \approx \frac{5}{3(12)}\left[0+400\left(\frac{5}{12}\right)\sqrt{125-\left(\frac{5}{12}\right)^3}+200\left(\frac{10}{12}\right)\sqrt{125-\left(\frac{10}{12}\right)^3}\right.$$
$$\left.+\,400\left(\frac{15}{12}\right)\sqrt{125-\left(\frac{15}{12}\right)^3}+\cdots+0\right]\approx 10{,}233.58 \text{ ft} \cdot \text{lb}$$

38. Simpson's Rule: $n = 6$

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{4}{3(6)}\left[1+\frac{4}{1+(1/6)^2}+\frac{2}{1+(2/6)^2}+\frac{4}{1+(3/6)^2}+\frac{2}{1+(4/6)^2}+\frac{4}{1+(5/6)^2}+\frac{1}{2}\right]$$
$$\approx 3.14159$$

39. (a) Trapezoidal: Area $\approx \frac{160}{2(8)}[0+2(50)+2(54)+2(82)+2(82)+2(73)+2(75)+2(80)+0] = 9920$ sq ft

(b) Simpson's: Area $\approx \frac{160}{3(8)}[0+4(50)+2(54)+4(82)+2(82)+4(73)+2(75)+4(80)+0] = 10{,}413\frac{1}{3}$ sq ft

40. Trapezoidal:

$$\int_0^2 f(x)\,dx \approx \frac{2}{2(8)}[4.32+2(4.36)+2(4.58)+2(5.79)+2(6.14)+2(7.25)+2(7.64)+2(8.08)+8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x)\,dx \approx \frac{2}{3(8)}[4.32+4(4.36)+2(4.58)+4(5.79)+2(6.14)+4(7.25)+2(7.64)+4(8.08)+8.14] \approx 12.592$$

41. Area $\approx \frac{1000}{2(10)}[125+2(125)+2(120)+2(112)+2(90)+2(90)+2(95)+2(88)+2(75)+2(35)] = 89{,}250$ sq m

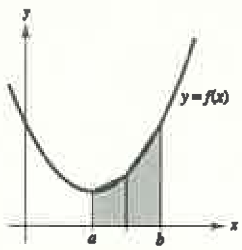
42. Area $\approx \frac{120}{2(12)}[75+2(81)+2(84)+2(76)+2(67)+2(68)+2(69)+2(72)+2(68)+2(56)+2(42)+2(23)+0]$

$= 7435$ sq m

43. $\int_0^t \sin\sqrt{x}\,dx = 2,\ n = 10$

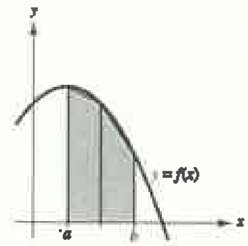
By trial and error, we obtain $t \approx 2.477$.

44. (a)



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

(b)



The Trapezoidal Rule underestimates the area if the graph of the integrand is concave down.

45. $L(x) = \int_1^x \frac{1}{t}\,dt$

(a) $L(1) = \int_1^1 \frac{1}{t}\,dt = 0$

(b) $L'(x) = \frac{1}{x}$

$L'(1) = 1$

(c) By trial and error, $x \approx 2.718$.

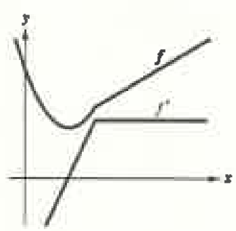
(d) $L(x_1x_2) = \int_1^{x_1x_2} \frac{1}{t}\,dt = \int_1^{x_1} \frac{1}{t}\,dt + \int_{x_1}^{x_1x_2} \frac{1}{t}\,dt$

But, this second integral is equal to $\int_1^{x_2}(1/t)\,dt$, by substitution $u = t/x_1$, $du = dt/x_1$. Hence,

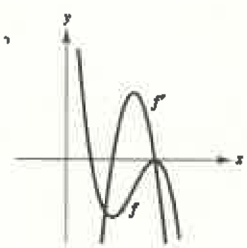
$L(x_1x_2) = L(x_1) + L(x_2)$.

Review Exercises for Chapter 4

1.



2.



3. $\int (2x^2 + x - 1)\,dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

4. $u = 3x$

$du = 3\,dx$

$$\int \frac{2}{\sqrt[3]{3x}}\,dx = \frac{2}{3}\int (3x)^{-1/3}(3)\,dx = (3x)^{2/3} + C$$

5. $\int \frac{x^3+1}{x^2}\,dx = \int\left(x + \frac{1}{x^2}\right)dx = \frac{1}{2}x^2 - \frac{1}{x} + C$

6. $\int \frac{x^3 - 2x^2 + 1}{x^2}\,dx = \int (x - 2 + x^{-2})\,dx$

$$= \frac{1}{2}x^2 - 2x - \frac{1}{x} + C$$

7. $\int (4x - 3\sin x)\,dx = 2x^2 + 3\cos x + C$

8. $\int (5\cos x - 2\sec^2 x)\,dx = 5\sin x - 2\tan x + C$

9. $f'(x) = -2x;\ (-1, 1)$

$$f(x) = \int -2x\,dx = -x^2 + C$$

When $x = -1$:

$$y = -1 + C = 1$$

$$C = 2$$

$$y = 2 - x^2$$

10. $f''(x) = 6(x - 1)$

$$f'(x) = \int 6(x - 1)\,dx = 3(x - 1)^2 + C_1$$

Since the slope of the tangent line at $(2, 1)$ is 3, it follows that $f'(2) = 3 + C_1 = 3$ when $C_1 = 0$.

$$f'(x) = 3(x - 1)^2$$

$$f(x) = \int 3(x - 1)^2\,dx = (x - 1)^3 + C_2$$

$$f(2) = 1 + C_2 = 1 \text{ when } C_2 = 0.$$

$$f(x) = (x - 1)^3$$

11. $a(t) = a$

$$v(t) = \int a\,dt = at + C_1$$

$$v(0) = 0 + C_1 = 0 \text{ when } C_1 = 0.$$

$$v(t) = at$$

$$s(t) = \int at\,dt = \frac{a}{2}t^2 + C_2$$

$\int at$

$at^2/2$

$$s(0) = 0 + C_2 = 0 \text{ when } C_2 = 0.$$

$$s(t) = \frac{a}{2}t^2$$

$$s(30) = \frac{a}{2}(30)^2 = 3600 \text{ or}$$

$$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2.$$

$$v(30) = 8(30) = 240 \text{ ft/sec}$$

12. $45 \text{ mph} = 66 \text{ ft/sec}$

$30 \text{ mph} = 44 \text{ ft/sec}$

$$a(t) = -a$$

$$v(t) = -at + 66 \text{ since } v(0) = 66 \text{ ft/sec.}$$

$$s(t) = -\frac{a}{2}t^2 + 66t \text{ since } s(0) = 0.$$

Solving the system

$$v(t) = -at + 66 = 44$$

$$s(t) = -\frac{a}{2}t^2 + 66t = 264$$

we obtain $t = 24/5$ and $a = 55/12$. We now solve $-(55/12)t + 66 = 0$ and get $t = 72/5$. Thus,

$$s\left(\frac{72}{5}\right) = -\frac{55/12}{2}\left(\frac{72}{5}\right)^2 + 66\left(\frac{72}{5}\right) \approx 475.2 \text{ ft.}$$

Stopping distance from 30 mph to rest is

$$475.2 - 264 = 211.2 \text{ ft.}$$

13. $a(t) = -32$

$$v(t) = -32t + 96$$

$$s(t) = -16t^2 + 96t$$

(a) $v(t) = -32t + 96 = 0$ when $t = 3$ sec.

(b) $s(3) = -144 + 288 = 144$ ft

(c) $v(t) = -32t + 96 = \frac{96}{2}$ when $t = \frac{3}{2}$ sec.

(d) $s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108$ ft

14. $a(t) = -9.8 \text{ m/sec}^2$

$$v(t) = -9.8t + v_0 = -9.8t + 40$$

$$s(t) = -4.9t^2 + 40t \quad (s(0) = 0)$$

(a) $v(t) = -9.8t + 40 = 0$ when $t = \frac{40}{9.8} \approx 4.08$ sec.

(b) $s(4.08) \approx 81.63$ m

(c) $v(t) = -9.8t + 40 = 20$ when $t = \frac{20}{9.8} \approx 2.04$ sec.

(d) $s(2.04) \approx 61.2$ m

15. (a) $\sum_{i=1}^{10}(2i - 1)$ (b) $\sum_{i=1}^{n} i^3$ (c) $\sum_{i=1}^{10}(4i + 2)$

16. $x_1 = 2, x_2 = -1, x_3 = 5, x_4 = 3, x_5 = 7$

(a) $\frac{1}{5}\sum_{i=1}^{5} x_i = \frac{1}{5}(2 - 1 + 5 + 3 + 7) = \frac{16}{5}$

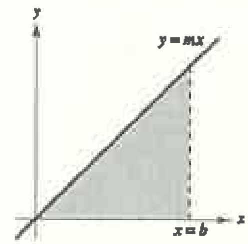
(b) $\sum_{i=1}^{5} \frac{1}{x_i} = \frac{1}{2} - 1 + \frac{1}{5} + \frac{1}{3} + \frac{1}{7} = \frac{37}{210}$

(c) $\sum_{i=1}^{5}(2x_i - x_i^2) = [2(2) - (2)^2] + [2(-1) - (-1)^2] + [2(5) - (5)^2] + [2(3) - (3)^2] + [2(7) - (7)^2] = -56$

(d) $\sum_{i=2}^{5}(x_i - x_{i-1}) = (-1 - 2) + [5 - (-1)] + (3 - 5) + (7 - 3) = 5$

17. (a) $S = m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{4b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3 + 4) = \frac{5mb^2}{8}$

$s = m(0)\left(\frac{b}{4}\right) + m\left(\frac{b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{2b}{4}\right)\left(\frac{b}{4}\right) + m\left(\frac{3b}{4}\right)\left(\frac{b}{4}\right) = \frac{mb^2}{16}(1 + 2 + 3) = \frac{3mb^2}{8}$



(b) $S(n) = \sum_{i=1}^{n} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=1}^{n}\left(\frac{mbi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2\sum_{i=1}^{n} i = \frac{mb^2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{mb^2(n+1)}{2n}$

$s(n) = \sum_{i=0}^{n-1} f\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = \sum_{i=0}^{n-1} m\left(\frac{bi}{n}\right)\left(\frac{b}{n}\right) = m\left(\frac{b}{n}\right)^2\sum_{i=0}^{n-1} i = \frac{mb^2}{n^2}\left(\frac{(n-1)n}{2}\right) = \frac{mb^2(n-1)}{2n}$

(c) Area $= \lim_{n\to\infty} \frac{mb^2(n+1)}{2n} = \lim_{n\to\infty} \frac{mb^2(n-1)}{2n} = \frac{1}{2}mb^2 = \frac{1}{2}(b)(mb) = \frac{1}{2}(\text{base})(\text{height})$

(d) $\int_0^b mx\,dx = \left[\frac{1}{2}mx^2\right]_0^b = \frac{1}{2}mb^2$

18. (a) $S(n) = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$

$= \sum_{i=1}^{n}\left(1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$

$= \sum_{i=1}^{n}\left(1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\left(\frac{2}{n}\right)$

$= 2 + \frac{12}{n^2}\sum_{i=1}^{n} i + \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$

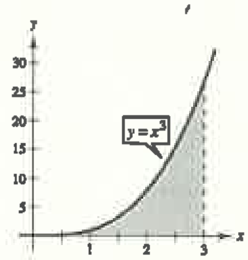
$= 2 + \frac{12}{n^2}[n(n+1)] + \frac{24}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{16}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$

$= 2 + 6\left(1 + \frac{1}{n}\right) + 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

Area $= \lim_{n\to\infty} S(n) = 2 + 6 + 8 + 4 = 20$

(b) $\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3$

$= \frac{81}{4} - \frac{1}{4} = 20$



19. (a) $\int_2^6 [f(x) + g(x)]\,dx = \int_2^6 f(x)\,dx + \int_2^6 g(x)\,dx = 10 + 3 = 13$

(b) $\int_2^6 [f(x) - g(x)]\,dx = \int_2^6 f(x)\,dx - \int_2^6 g(x)\,dx = 10 - 3 = 7$

(c) $\int_2^6 [2f(x) - 3g(x)]\,dx = 2\int_2^6 f(x)\,dx - 3\int_2^6 g(x)\,dx = 2(10) - 3(3) = 11$

(d) $\int_2^6 5f(x)\,dx = 5\int_2^6 f(x)\,dx = 5(10) = 50$

20. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_4^4 f(x)\,dx = 0$

(d) $\int_3^6 -10f(x)\,dx = -10\int_3^6 f(x)\,dx = -10(-1) = 10$

21. $\int (x^2+1)^3\,dx = \int (x^6 + 3x^4 + 3x^2 + 1)\,dx$

$= \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$

22. $\int \left(x + \frac{1}{x}\right)^2 dx = \int (x^2 + 2 + x^{-2})\,dx$

$= \frac{x^3}{3} + 2x - \frac{1}{x} + C$

23. $u = x^3 + 3,\ du = 3x^2\,dx$

$\int \frac{x^2}{\sqrt{x^3+3}}\,dx = \int (x^3+3)^{-1/2}x^2\,dx = \frac{1}{3}\int (x^3+3)^{-1/2}\,3x^2\,dx = \frac{2}{3}(x^3+3)^{1/2} + C$

24. $u = x^3 + 3,\ du = 3x^2\,dx$

$\int x^2\sqrt{x^3+3}\,dx = \frac{1}{3}\int (x^3+3)^{1/2}\,3x^2\,dx = \frac{2}{9}(x^3+3)^{3/2} + C$

25. $u = 1 - 3x^2,\ du = -6x\,dx$

$\int x(1-3x^2)^4\,dx = -\frac{1}{6}\int (1-3x^2)^4(-6x\,dx) = -\frac{1}{30}(1-3x^2)^5 + C = \frac{1}{30}(3x^2-1)^5 + C$

26. $u = x^2 + 6x - 5,\ du = (2x+6)\,dx$

$\int \frac{x+3}{(x^2+6x-5)^2} = \frac{1}{2}\int \frac{2x+6}{(x^2+6x-5)^2}\,dx = \frac{-1}{2}(x^2+6x-5)^{-1} + C = \frac{-1}{2(x^2+6x-5)} + C$

27. $\int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

28. $\int x\sin 3x^2\,dx = \frac{1}{6}\int (\sin 3x^2)(6x)\,dx = -\frac{1}{6}\cos 3x^2 + C$

29. $\int \frac{\sin\theta}{\sqrt{1-\cos\theta}}\,d\theta = \int (1-\cos\theta)^{-1/2}\sin\theta\,d\theta = 2(1-\cos\theta)^{1/2} + C = 2\sqrt{1-\cos\theta} + C$

30. $\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int (\sin x)^{-1/2}\cos x\,dx = 2(\sin x)^{1/2} + C = 2\sqrt{\sin x} + C$

31. $\int \tan^n x\sec^2 x\,dx = \frac{\tan^{n+1}x}{n+1} + C,\ n \neq -1$

32. $\int \sec 2x\tan 2x\,dx = \frac{1}{2}\int (\sec 2x\tan 2x)(2)\,dx = \frac{1}{2}\sec 2x + C$

33. $\int (1+\sec\pi x)^2\sec\pi x\tan\pi x\,dx = \frac{1}{\pi}\int (1+\sec\pi x)^2(\pi\sec\pi x\tan\pi x)\,dx = \frac{1}{3\pi}(1+\sec\pi x)^3 + C$

34. $\displaystyle\int \cot^4 \alpha \csc^2 \alpha\, d\alpha = -\int (\cot \alpha)^4(-\csc^2 \alpha)\, d\alpha$

$\displaystyle = -\frac{1}{5}\cot^5 \alpha + C$

35. $\displaystyle\int_0^4 (2 + x)\, dx = \left[2x + \frac{x^2}{2}\right]_0^4 = 8 + \frac{16}{2} = 16$

36. $\displaystyle\int_{-1}^1 (t^2 + 2)\, dt = \left[\frac{t^3}{3} + 2t\right]_{-1}^1 = \frac{14}{3}$

37. $\displaystyle\int_{-1}^1 (4t^3 - 2t)\, dt = \left[t^4 - t^2\right]_{-1}^1 = 0$

38. $\displaystyle\int_3^6 \frac{x}{3\sqrt{x^2 - 8}}\, dx = \frac{1}{6}\int_3^6 (x^2 - 8)^{-1/2}(2x)\, dx = \left[\frac{1}{3}(x^2 - 8)^{1/2}\right]_3^6 = \frac{1}{3}(2\sqrt{7} - 1)$

39. $\displaystyle\int_0^3 \frac{1}{\sqrt{1 + x}}\, dx = \int_0^3 (1 + x)^{-1/2}\, dx = \left[2(1 + x)^{1/2}\right]_0^3 = 4 - 2 = 2$

40. $\displaystyle\int_0^1 x^2(x^3 + 1)^3\, dx = \frac{1}{3}\int_0^1 (x^3 + 1)^3(3x^2)\, dx = \frac{1}{12}\left[(x^3 + 1)^4\right]_0^1 = \frac{1}{12}(16 - 1) = \frac{5}{4}$

41. $\displaystyle\int_4^9 x\sqrt{x}\, dx = \int_4^9 x^{3/2}\, dx = \left[\frac{2}{5}x^{5/2}\right]_4^9 = \frac{2}{5}\left[\left(\sqrt{9}\right)^5 - \left(\sqrt{4}\right)^5\right] = \frac{2}{5}(243 - 32) = \frac{422}{5}$

42. $\displaystyle\int_1^2 \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx = \int_1^2 (x^{-2} - x^{-3})\, dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_1^2 = \left(-\frac{1}{2} + \frac{1}{8}\right) - \left(-1 + \frac{1}{2}\right) = \frac{1}{8}$

43. $u = 1 - y,\ y = 1 - u,\ dy = -du$

When $y = 0$, $u = 1$. When $y = 1$, $u = 0$.

$$2\pi\int_0^1 (y + 1)\sqrt{1 - y}\, dy = 2\pi\int_1^0 -[(1 - u) + 1]\sqrt{u}\, du$$

$$= 2\pi\int_1^0 (u^{3/2} - 2u^{1/2})\, du = 2\pi\left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right]_1^0 = \frac{28\pi}{15}$$

44. $u = x + 1,\ x = u - 1,\ dx = du$

When $x = -1$, $u = 0$. When $x = 0$, $u = 1$.

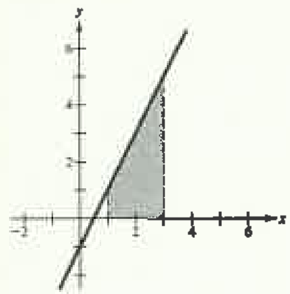
$$2\pi\int_{-1}^0 x^2\sqrt{x + 1}\, dx = 2\pi\int_0^1 (u - 1)^2\sqrt{u}\, du$$

$$= 2\pi\int_0^1 (u^{5/2} - 2u^{3/2} + u^{1/2})\, du = 2\pi\left[\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 = \frac{32\pi}{105}$$

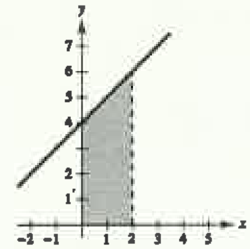
45. $\displaystyle\int_0^\pi \cos\left(\frac{x}{2}\right) dx = 2\int_0^\pi \cos\left(\frac{x}{2}\right)\frac{1}{2}\, dx = \left[2\sin\left(\frac{x}{2}\right)\right]_0^\pi = 2$

46. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin 2x\, dx = 0$ since $\sin 2x$ is an odd function.

47. $\int_1^3 (2x - 1)\,dx = \Big[x^2 - x\Big]_1^3 = 6$



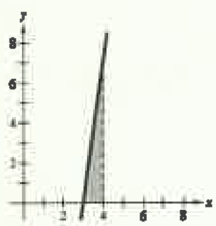
48. $\int_0^2 (x + 4)\,dx = \left[\frac{x^2}{2} + 4x\right]_0^2 = 10$



49. $\int_3^4 (x^2 - 9)\,dx = \left[\frac{x^3}{3} - 9x\right]_3^4$

$= \left(\frac{64}{3} - 36\right) - (9 - 27)$

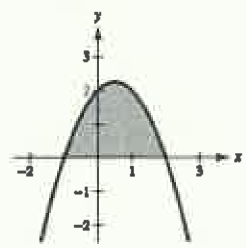
$= \frac{64}{3} - \frac{54}{3} = \frac{10}{3}$



50. $\int_{-1}^2 (-x^2 + x + 2)\,dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^2$

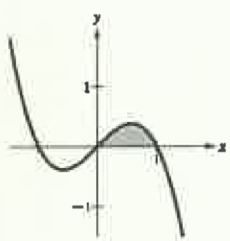
$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$

$= \frac{10}{3} + \frac{7}{6} = \frac{9}{2}$



51. $\int_0^1 (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$

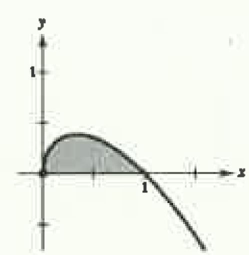
$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$



52. $\int_0^1 = \sqrt{x}(1 - x)\,dx = (x^{1/2} - x^{3/2})\,dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$

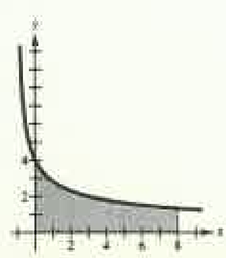
$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$



53. $\int_0^8 \frac{4}{\sqrt{x + 1}}\,dx = 4\int_0^8 (x + 1)^{-1/2}\,dx$

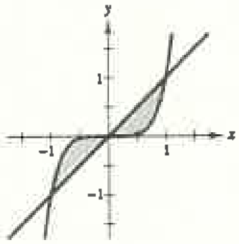
$= \Big[8\sqrt{x + 1}\Big]_0^8$

$= 8(3 - 1) = 16$

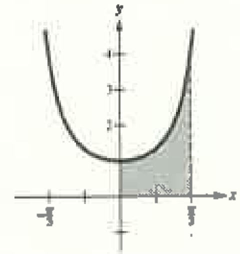


54. $\displaystyle\int_{-1}^{0}(x^5 - x)\,dx + \int_0^1 (x - x^5)\,dx = 2\int_0^1 (x - x^5)\,dx$

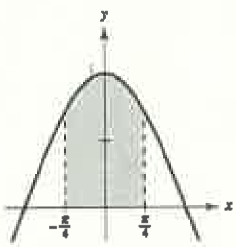
$\displaystyle = 2\left[\frac{x^2}{2} - \frac{x^6}{6}\right]_0^1 = \frac{2}{3}$



55. $\displaystyle\int_0^{\pi/3} \sec^2 x\,dx = \Big[\tan x\Big]_0^{\pi/3} = \sqrt{3}$



56. $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\,dx = 2\int_0^{\pi/4} \cos x\,dx = \Big[2\sin x\Big]_0^{\pi/4} = \sqrt{2}$



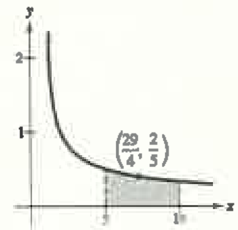
57. $\displaystyle\frac{1}{10-5}\int_5^{10} \frac{1}{\sqrt{x-1}}\,dx = \left[\frac{2}{5}\sqrt{x-1}\right]_5^{10} = \frac{2}{5}$

$\displaystyle\frac{1}{\sqrt{x-1}} = \frac{2}{5}$

$\displaystyle\sqrt{x-1} = \frac{5}{2}$

$\displaystyle x - 1 = \frac{25}{4}$

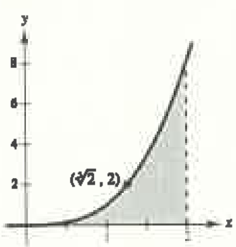
$\displaystyle x = \frac{29}{4}$



58. $\displaystyle\frac{1}{2-0}\int_0^2 x^3\,dx = \left[\frac{x^4}{8}\right]_0^2 = 2$

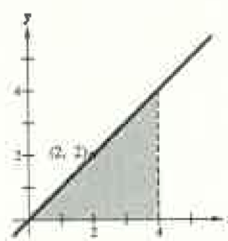
$x^3 = 2$

$x = \sqrt[3]{2}$



59. $\displaystyle\frac{1}{4-0}\int_0^4 x\,dx = \left[\frac{x^2}{8}\right]_0^4 = 2$

$x = 2$



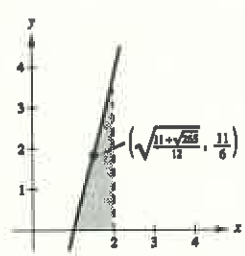
60. $\displaystyle\frac{1}{2-1}\int_1^2 \left(x^2 - \frac{1}{x^2}\right)dx = \left[\frac{x^3}{3} + \frac{1}{x}\right]_1^2 = \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) = \frac{11}{6}$

$\displaystyle x^2 - \frac{1}{x^2} = \frac{11}{6}$

$6x^4 - 6 = 11x^2$

$\displaystyle 6x^4 - 11x^2 - 6 = 0$ when $\displaystyle x^2 = \frac{11 \pm \sqrt{265}}{12}$.

In the interval $[1, 2]$: $\displaystyle x = \sqrt{\frac{11 + \sqrt{265}}{12}} \approx 1.508.$



61. Trapezoidal Rule ($n = 4$): $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{8}\left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.257$

Simpson's Rule ($n = 4$): $\displaystyle\int_1^2 \frac{1}{1+x^3}\,dx \approx \frac{1}{12}\left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3}\right] \approx 0.254$

Graphing utility: 0.254

62. Trapezoidal Rule ($n = 4$): $\displaystyle\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{8}\left[0 + \frac{2(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{2(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.172$

Simpson's Rule ($n = 4$): $\displaystyle\int_0^1 \frac{x^{3/2}}{3-x^2}\,dx \approx \frac{1}{12}\left[0 + \frac{4(1/4)^{3/2}}{3-(1/4)^2} + \frac{2(1/2)^{3/2}}{3-(1/2)^2} + \frac{4(3/4)^{3/2}}{3-(3/4)^2} + \frac{1}{2}\right] \approx 0.166$

Graphing utility: 0.166

63. (a) $\displaystyle C = 0.1\int_8^{20}\left[12\sin\frac{\pi(t-8)}{12}\right]dt = \left[-\frac{14.4}{\pi}\cos\frac{\pi(t-8)}{12}\right]_8^{20} = \frac{-14.4}{\pi}(-1-1) \approx \$9.17$

(b) $\displaystyle C = 0.1\int_{10}^{18}\left[12\sin\frac{\pi(t-8)}{12} - 6\right]dt = \left[-\frac{14.4}{\pi}\cos\frac{\pi(t-8)}{12} - 0.6t\right]_{10}^{18}$

$$= \left[\frac{-14.4}{\pi}\left(\frac{-\sqrt{3}}{2}\right) - 10.8\right] - \left[\frac{-14.4}{\pi}\left(\frac{\sqrt{3}}{2}\right) - 6\right] \approx \$3.14$$

Savings $\approx 9.17 - 3.14 = \$6.03$.

64. $\displaystyle \frac{1}{365}\int_0^{365} 100{,}000\left[1 + \sin\frac{2\pi(t-60)}{365}\right]dt = \frac{100{,}000}{365}\left[t - \frac{365}{2\pi}\cos\frac{2\pi(t-60)}{365}\right]_0^{365} = 100{,}000$ lbs.

65. $\displaystyle V = \int_0^3 0.85\sin\frac{\pi t}{3}\,dt = -\frac{3}{\pi}\left[0.85\cos\frac{\pi t}{3}\right]_0^3 = -\frac{2.55}{\pi}(-1-1) = \frac{5.1}{\pi} \approx 1.6234$ liters

66. $\displaystyle \int_0^2 1.75\sin\frac{\pi t}{2}\,dt = -\frac{2}{\pi}\left[1.75\cos\frac{\pi t}{2}\right]_0^2 = -\frac{2}{\pi}(1.75)(-1-1) = \frac{7}{\pi} \approx 2.2282$ liters

Increase is

$\displaystyle \frac{7}{\pi} - \frac{5.1}{\pi} = \frac{1.9}{\pi} \approx 0.6048$ liters.

67. $p = 1.20 + 0.04t$

$\displaystyle C = \frac{15{,}000}{M}\int_t^{t+1} p\,ds$

(a) 2000 corresponds to $t = 10$.

$\displaystyle C = \frac{15{,}000}{M}\int_{10}^{11}[1.20 + 0.04t]\,dt$

$\displaystyle = \frac{15{,}000}{M}\left[1.20t + 0.02t^2\right]_{10}^{11} = \frac{24{,}300}{M}$

(b) 2005 corresponds to $t = 15$.

$\displaystyle C = \frac{15{,}000}{M}\left[1.20t + 0.02t^2\right]_{15}^{16} = \frac{27{,}300}{M}$

68. $u = 1 - x, x = 1 - u, dx = -du$

When $x = a, u = 1 - a$. When $x = b, u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{15}{4}x\sqrt{1-x}\,dx = \frac{15}{4}\int_{1-a}^{1-b} -(1-u)\sqrt{u}\,du$$

$$= \frac{15}{4}\int_{1-a}^{1-b}(u^{3/2} - u^{1/2})\,du = \frac{15}{4}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_{1-a}^{1-b} = \frac{15}{4}\left[\frac{2u^{3/2}}{15}(3u-5)\right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_a^b$$

(a) $P_{0.50,\,0.75} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_{0.50}^{0.75} = 0.353 = 35.3\%$

(b) $P_{0,b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2)\right]_0^b = -\frac{(1-b)^{3/2}}{2}(3b+2) + 1 = 0.5$

$(1-b)^{3/2}(3b+2) = 1$

$b \approx 0.586 \approx 58.6\%$

69. $u = 1 - x, x = 1 - u, dx = -du$

When $x = a, u = 1 - a$. When $x = b, u = 1 - b$.

$$P_{a,b} = \int_a^b \frac{1155}{32}x^3(1-x)^{3/2}\,dx = \frac{1155}{32}\int_{1-a}^{1-b} -(1-u)^3 u^{3/2}\,du$$

$$= \frac{1155}{32}\int_{1-a}^{1-b}(u^{9/2} - 3u^{7/2} + 3u^{5/2} - u^{3/2})\,du = \frac{1155}{32}\left[\frac{2}{11}u^{11/2} - \frac{2}{3}u^{9/2} + \frac{6}{7}u^{7/2} - \frac{2}{5}u^{5/2}\right]_{1-a}^{1-b}$$

$$= \frac{1155}{32}\left[\frac{2u^{5/2}}{1155}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{1-a}^{1-b}$$

(a) $P_{0,\,0.25} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_1^{0.75} \approx 0.025 = 2.5\%$

(b) $P_{0.5,\,1} = \left[\frac{u^{5/2}}{16}(105u^3 - 385u^2 + 495u - 231)\right]_{0.5}^{0} \approx 0.736 = 73.6\%$

70. False. This property only holds for **constants.**

71. False

$$\int -\frac{1}{x^2}\,dx = \frac{1}{x} + C \text{ or } \frac{d}{dx}\left[-\frac{1}{x^2}\right] \neq \frac{1}{x}.$$

72. True. If $f(x) = -f(-x)$, then $f(-x) = -f(x)$ and $f(x)$ is odd.

73. True

$$\frac{1}{2\pi - 0}\int_0^{2\pi} \sin x\,dx = \left[-\frac{1}{2\pi}\cos x\right]_0^{2\pi} = 0$$

74. False

$$\frac{d}{dx}[\sec^2 x] \neq \tan x$$

CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function and Differentiation

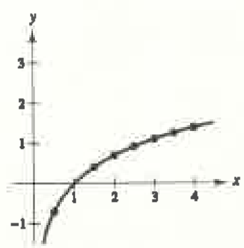
Solutions to Exercises

1. Simpson's Rule: $n = 10$

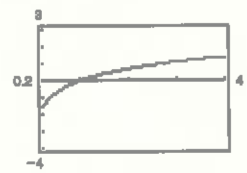
x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t}\,dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t}\,dt = -\int_{0.5}^1 \frac{1}{t}\,dt$

2. (a)



(b)



The graphs are identical.

3. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

4. $f(x) = -\ln x$

Reflection in the x-axis

Matches (d)

5. $f(x) = \ln (x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

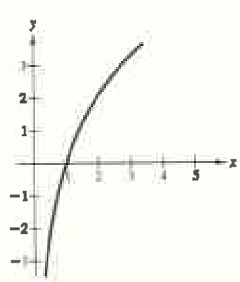
6. $f(x) = -\ln (-x)$

Reflection in the y-axis and the x-axis

Matches (c)

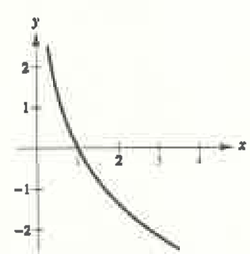
7. $f(x) = 3 \ln x$

Domain: $x > 0$



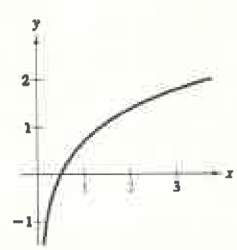
8. $f(x) = -2 \ln x$

Domain: $x > 0$



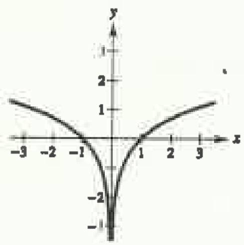
9. $f(x) = \ln 2x$

Domain: $x > 0$



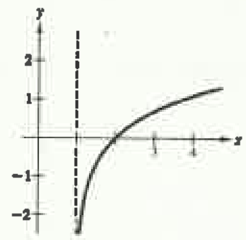
10. $f(x) = \ln|x|$

Domain: $x \neq 0$



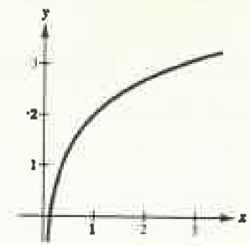
11. $f(x) = \ln(x - 1)$

Domain: $x > 1$



12. $g(x) = 2 + \ln x$

Domain: $x > 0$



13. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

14. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

15. $\ln \frac{2}{3} = \ln 2 - \ln 3$

16. $\ln \frac{1}{5} = \ln 1 - \ln 5 = -\ln 5$

17. $\ln \dfrac{xy}{z} = \ln x + \ln y - \ln z$

18. $\ln xyz = \ln x + \ln y + \ln z$

19. $\ln \sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2} \ln 2$

20. $\ln \sqrt{a - 1} = \ln(a - 1)^{1/2} = \left(\frac{1}{2}\right) \ln(a - 1)$

21. $\ln\left(\dfrac{x^2 - 1}{x^3}\right)^3 = 3[\ln(x^2 - 1) - \ln x^3]$

$= 3[\ln(x + 1) + \ln(x - 1) - 3 \ln x]$

22. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

23. $\ln z(z - 1)^2 = \ln z + \ln(z - 1)^2$

$= \ln z + 2 \ln(z - 1)$

24. $\ln \dfrac{1}{e} = \ln 1 - \ln e = -1$

25. $\ln(x - 2) - \ln(x + 2) = \ln \dfrac{x - 2}{x + 2}$

26. $3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$

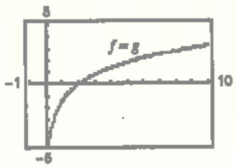
$= \ln \dfrac{x^3 y^2}{z^4}$

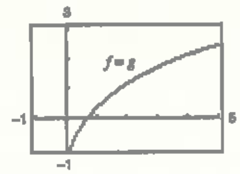
27. $\dfrac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)] = \dfrac{1}{3} \ln \dfrac{x(x + 3)^2}{x^2 - 1} = \ln \sqrt[3]{\dfrac{x(x + 3)^2}{x^2 - 1}}$

28. $2[\ln x - \ln(x + 1) - \ln(x - 1)] = 2 \ln \dfrac{x}{(x + 1)(x - 1)} = \ln\left(\dfrac{x}{x^2 - 1}\right)^2$

29. $2 \ln 3 - \dfrac{1}{2} \ln(x^2 + 1) = \ln 9 - \ln \sqrt{x^2 + 1} = \ln \dfrac{9}{\sqrt{x^2 + 1}}$

30. $\dfrac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)] = \dfrac{3}{2} \ln \dfrac{x^2 + 1}{(x + 1)(x - 1)} = \ln \sqrt{\left(\dfrac{x^2 + 1}{x^2 - 1}\right)^3}$

31. 

32. 

33. $\lim_{x \to 3^+} \ln(x - 3) = -\infty$

34. $\lim_{x \to 6^-} \ln(6 - x) = -\infty$

35. $\lim_{x \to 2^-} \ln[x^2(3 - x)] = \ln 4 \approx 1.3863$

36. $\lim_{x \to 5^+} \ln \dfrac{x}{\sqrt{x - 4}} = \ln 5 \approx 1.6094$

37. $y = \ln x^3 = 3 \ln x$

$y' = \dfrac{3}{x}$

At $(1, 0)$, $y' = 3$.

38. $y = \ln x^{3/2} = \dfrac{3}{2} \ln x$

$y' = \dfrac{3}{2x}$

At $(1, 0)$, $y' = \frac{3}{2}$.

39. $y = \ln x^2 = 2 \ln x$

$y' = \dfrac{2}{x}$

At $(1, 0)$, $y' = 2$.

40. $y = \ln x^{1/2} = \dfrac{1}{2} \ln x$

$y' = \dfrac{1}{2x}$

At $(1, 0)$, $y' = \frac{1}{2}$.

41. $g(x) = \ln x^2 = 2 \ln x$

$g'(x) = \dfrac{2}{x}$

42. $h(x) = \ln(x^2 + 3)$

$h'(x) = \dfrac{2x}{x^2 + 3}$

43. $y = (\ln x)^4$

$$\frac{dy}{dx} = 4(\ln x)^3\left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$$

44. $y = x \ln x$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

45. $y = \ln x\sqrt{x^2 - 1} = \ln x + \dfrac{1}{2}\ln(x^2 - 1)$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2 - 1}\right) = \frac{2x^2 - 1}{x(x^2 - 1)}$$

46. $y = \ln\sqrt{x^2 - 4} = \dfrac{1}{2}\ln(x^2 - 4)$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{2x}{x^2 - 4}\right) = \frac{x}{x^2 - 4}$$

47. $f(x) = \ln \dfrac{x}{x^2 + 1} = \ln x - \ln(x^2 + 1)$

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x(x^2 + 1)}$$

48. $f(x) = \ln \dfrac{x}{x + 1} = \ln x - \ln(x + 1)$

$$f'(x) = \frac{1}{x} - \frac{1}{x + 1} = \frac{1}{x^2 + x}$$

49. $g(t) = \dfrac{\ln t}{t^2}$

$$g'(t) = \frac{t^2(1/t) - 2t \ln t}{t^4} = \frac{1 - 2 \ln t}{t^3}$$

50. $h(t) = \dfrac{\ln t}{t}$

$$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$$

51. $y = \ln(\ln x^2)$

$$\frac{dy}{dx} = \frac{1}{\ln x^2}\frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}$$

52. $y = \ln(\ln x)$

$$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

53. $y = \ln\sqrt{\dfrac{x+1}{x-1}} = \dfrac{1}{2}[\ln(x+1) - \ln(x-1)]$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{1}{1-x^2}$$

54. $y = \ln\sqrt{\dfrac{x-1}{x+1}} = \dfrac{1}{2}[\ln(x-1) - \ln(x+1)]$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{1}{x^2-1}$$

55. $f(x) = \ln\dfrac{\sqrt{4+x^2}}{x} = \dfrac{1}{2}\ln(4+x^2) - \ln x$

$$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} = \frac{-4}{x(x^2+4)}$$

56. $f(x) = \ln\left(x + \sqrt{4+x^2}\right)$

$$f'(x) = \frac{1}{x+\sqrt{4+x^2}}\left(1 + \frac{x}{\sqrt{4+x^2}}\right)$$
$$= \frac{1}{\sqrt{4+x^2}}$$

57. $y = \dfrac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = \frac{-x\left(x/\sqrt{x^2+1}\right) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2}$$

58. $y = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2+\sqrt{x^2+4}}{x}\right) = \dfrac{-\sqrt{x^2+4}}{2x^2} - \dfrac{1}{4}\ln\left(2+\sqrt{x^2+4}\right) + \dfrac{1}{4}\ln x$

$$\frac{dy}{dx} = \frac{-2x^2\left(x/\sqrt{x^2+4}\right) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4}\left(\frac{1}{2+\sqrt{x^2+4}}\right)\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$

Note that:

$$\frac{1}{2+\sqrt{x^2+4}} = \frac{1}{2+\sqrt{x^2+4}}\cdot\frac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \frac{2-\sqrt{x^2+4}}{-x^2}$$

Hence, $$\frac{dy}{dx} = \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4}\frac{\left(2-\sqrt{x^2+4}\right)}{-x^2}\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x}$$
$$= \frac{-1 + (1/2)\left(2-\sqrt{x^2+4}\right)}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x}$$
$$= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3}$$

59. $y = \ln|\sin x|$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

60. $y = \ln|\sec x|$

$$\frac{dy}{dx} = \frac{\sec x\tan x}{\sec x} = \tan x$$

61. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

62. $y = \ln|\sec x + \tan x|$

$$\frac{dy}{dx} = \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

63. $y = \ln\left|\dfrac{-1+\sin x}{2+\sin x}\right|$

$$= \ln|-1+\sin x| - \ln|2+\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1+\sin x} - \frac{\cos x}{2+\sin x}$$

$$= \frac{3\cos x}{(\sin x - 1)(\sin x + 2)}$$

64. $y = \ln\sqrt{1+\sin^2 x} = \dfrac{1}{2}\ln(1+\sin^2 x)$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)\frac{2\sin x\cos x}{1+\sin^2 x} = \frac{\sin x\cos x}{1+\sin^2 x}$$

65. $f(x) = \sin 2x \ln x^2 = 2\sin 2x \ln x$

$$f'(x) = (2\sin 2x)\left(\frac{1}{x}\right) + 4\cos 2x \ln x$$

$$= \frac{2}{x}(\sin 2x + 2x\cos 2x \ln x)$$

$$= \frac{2}{x}(\sin 2x + x\cos 2x \ln x^2)$$

66. $g(x) = \displaystyle\int_1^{\ln x} (t^2+3)\,dt$

$$g'(x) = [(\ln x)^2 + 3]\frac{d}{dx}(\ln x) = \frac{(\ln x)^2+3}{x}$$

(Second Fundamental Theorem of Calculus)

67. (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

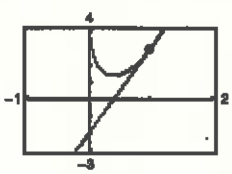
When $x = 1$, $\dfrac{dy}{dx} = 5$.

Tangent line: $y - 3 = 5(x-1)$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



68. (a) $y = 4 - x^2 - \ln\left(\dfrac{1}{2}x + 1\right), \quad (0, 4)$

$$\frac{dy}{dx} = -2x - \frac{1}{(1/2)x+1}\left(\frac{1}{2}\right)$$

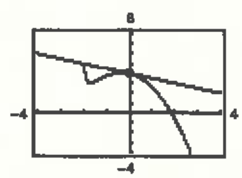
$$= -2x - \frac{1}{x+2}$$

When $x = 0$, $\dfrac{dy}{dx} = -\dfrac{1}{2}$.

Tangent line: $y - 4 = -\dfrac{1}{2}(x-0)$

$$y = -\frac{1}{2}x + 4$$

(b)



69. $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

70. $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y+5xy}{x}\right)$$

71. $y = 2(\ln x) + 3$

$y' = \dfrac{2}{x}$

$y'' = -\dfrac{2}{x^2}$

$xy'' + y' = x\left(-\dfrac{2}{x^2}\right) + \dfrac{2}{x} = 0$

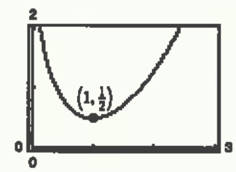
72. $y = x(\ln x) - 4x$

$y' = x\left(\dfrac{1}{x}\right) + \ln x - 4 = -3 + \ln x$

$(x + y) - xy' = x + x\ln x - 4x - x(-3 + \ln x)$

$= 0$

73. $y = \dfrac{x^2}{2} - \ln x$



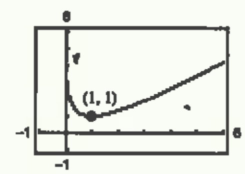
Domain: $x > 0$

$y' = x - \dfrac{1}{x} = \dfrac{(x + 1)(x - 1)}{x} = 0$ when $x = 1$.

$y'' = 1 + \dfrac{1}{x^2} > 0$

Relative minimum: $\left(1, \frac{1}{2}\right)$

74. $y = x - \ln x$



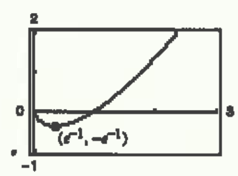
Domain: $x > 0$

$y' = 1 - \dfrac{1}{x} = 0$ when $x = 1$.

$y'' = \dfrac{1}{x^2} > 0$

Relative minimum: $(1, 1)$

75. $y = x \ln x$



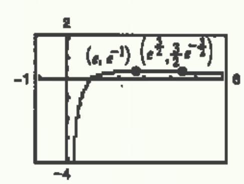
Domain: $x > 0$

$y' = x\left(\dfrac{1}{x}\right) + \ln x = 1 + \ln x = 0$ when $x = e^{-1}$.

$y'' = \dfrac{1}{x} > 0$

Relative minimum: $(e^{-1}, -e^{-1})$

76. $y = \dfrac{\ln x}{x}$



Domain: $x > 0$

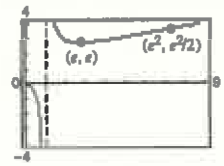
$y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

$y'' = \dfrac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \dfrac{2(\ln x) - 3}{x^3} = 0$ when $x = e^{3/2}$.

Relative maximum: (e, e^{-1})

Point of inflection: $\left(e^{3/2}, \frac{3}{2}e^{-3/2}\right)$

77. $y = \dfrac{x}{\ln x}$



Domain: $0 < x < 1, x > 1$

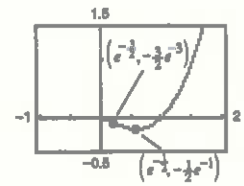
$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x)\ln x}{(\ln x)^4} = \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $(e^2, e^2/2)$

78. $y = x^2 \ln x$



Domain: $x > 0$

$$y' = x^2\left(\frac{1}{x}\right) + 2x(\ln x) = x(1 + 2\ln x) = 0 \text{ when } x = e^{-1/2}.$$

$$y'' = x\left(\frac{2}{x}\right) + (1 + 2\ln x) = 3 + 2(\ln x) = 0 \text{ when } x = e^{-3/2}.$$

Relative minimum: $\left(e^{-1/2}, -\frac{1}{2}e^{-1}\right)$

Point of inflection: $\left(e^{-3/2}, -\frac{3}{2}e^{-3}\right)$

79. $f(x) = \ln x, \quad f(1) = 0$

$f'(x) = \dfrac{1}{x}, \quad f'(1) = 1$

$f''(x) = -\dfrac{1}{x^2}, \quad f''(1) = -1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \dfrac{1}{2}f''(1)(x - 1)^2$

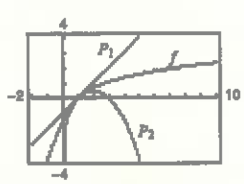
$\qquad = (x - 1) - \dfrac{1}{2}(x - 1)^2, \quad P_2(1) = 0$

$P_1'(x) = 1, \quad P_1'(1) = 1$

$P_2'(x) = 1 - (x - 1) = 2 - x, \quad P_2'(1) = 1$

$P_2''(x) = -1, \quad P_2''(1) = -1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



80. $f(x) = x\ln x, \quad f(1) = 0$

$f'(x) = 1 + \ln x, \quad f(1) = 1$

$f''(x) = \dfrac{1}{x}, \quad f''(1) = 1$

$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$

$P_2(x) = f(1) + f'(1)(x - 1) + \dfrac{1}{2}f''(1)(x - 1)^2$

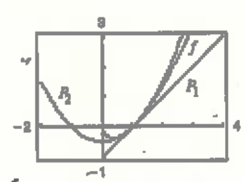
$\qquad = (x - 1) + \dfrac{1}{2}(x - 1)^2, \quad P_2(1) = 0$

$P_1'(x) = 1, \quad P_1'(1) = 1$

$P_2'(x) = 1 + (x - 1) = x, \quad P_2'(1) = 1$

$P_2''(x) = x, \quad P_2''(1) = 1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



81. Find x such that $\ln x = -x$.

$$f(x) = (\ln x) + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{1 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	−0.1931	−0.0076	−0.0001

Approximate root: $x = 0.567$

82. Find x such that $\ln x = 3 - x$.

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{4 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	−0.3069	−0.0049	0.0000

Approximate root: $x = 2.208$

83.
$$y = x\sqrt{x^2 - 1}$$
$$\ln y = \ln x + \frac{1}{2}\ln(x^2 - 1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$
$$\frac{dy}{dx} = y\left[\frac{2x^2 - 1}{x(x^2 - 1)}\right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

84.
$$y = \sqrt{(x-1)(x-2)(x-3)}$$
$$\ln y = \frac{1}{2}[\ln(x-1) + \ln(x-2) + \ln(x-3)]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}\right]$$
$$= \frac{1}{2}\left[\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\right]$$
$$\frac{dy}{dx} = \frac{3x^2 - 12x + 11}{2y}$$
$$= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}}$$

85.
$$y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$$
$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x-1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$
$$\frac{dy}{dx} = y\left[\frac{3x^2 - 15x + 8}{2x(3x-2)(x-1)}\right]$$
$$= \frac{3x^3 - 15x^2 + 8x}{2(x-1)^3\sqrt{3x-2}}$$

86.
$$y = \sqrt[3]{\frac{x^2+1}{x^2-1}}$$
$$\ln y = \frac{1}{3}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{3}\left[\frac{2x}{x^2+1} - \frac{1}{x+1} - \frac{1}{x-1}\right]$$
$$\frac{dy}{dx} = \frac{-4xy}{3(x^4-1)} = -\frac{4x}{3(x^4-1)}\sqrt[3]{\frac{x^2+1}{x^2-1}}$$
$$= \frac{-4x}{3(x^2+1)^{2/3}(x^2-1)^{4/3}}$$

87.
$$y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$
$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]$$
$$= \frac{y}{2}\left[\frac{4x^2 + 4x - 2}{x(x^2-1)}\right] = \frac{(2x^2 + 2x - 1)\sqrt{x-1}}{(x+1)^{3/2}}$$

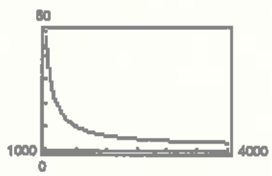
88.
$$y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$$
$$\ln y = \ln(x+1) + \ln(x+2) - \ln(x-1) - \ln(x-2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x-1} - \frac{1}{x-2}$$
$$\frac{dy}{dx} = y\left[\frac{-2}{x^2-1} + \frac{-4}{x^2-4}\right] = y\left[\frac{-6x^2 + 12}{(x^2-1)(x^2-4)}\right]$$
$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{-6(x^2-2)}{(x+1)(x-1)(x+2)(x-2)}$$
$$= \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$$

89. $$\beta = 10\log_{10}\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}\ln\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}[\ln I + 16\ln 10]$$

$$\beta(10^{-10}) = \frac{10}{\ln 10}[\ln 10^{-10} + 16\ln 10] = \frac{10}{\ln 10}[-10\ln 10 + 16\ln 10] = \frac{10}{\ln 10}[6\ln 10] = 60 \text{ decibels}$$

90. $t = \dfrac{5.315}{-6.7968 + \ln x}$, $1000 < x$

(a)



(b) $t(1167.41) \approx 20$ years

$T = (1167.41)(20)(12) = \$280{,}178.40$

(c) $t(1068.45) \approx 30$ years

$T = (1068.45)(30)(12) = \$384{,}642.00$

(d) $\dfrac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2}\left(\dfrac{1}{x}\right)$

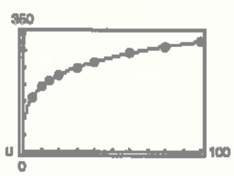
$= -\dfrac{5.315}{x(-6.7968 + \ln x)^2}$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

(e) There are two obvious benefits to paying a higher monthly payment:

1. The term is lower
2. The total amount paid is lower.

91. (a)

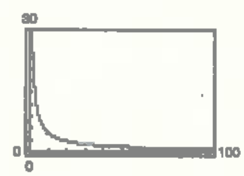


(b) $T'(p) = \dfrac{34.96}{p} + \dfrac{3.955}{\sqrt{p}}$

$T'(10) \approx 4.75$ deg/lb/in^2

$T'(70) \approx 0.97$ deg/lb/in^2

(c)



$\lim_{p\to\infty} T'(p) = 0$

92. $g(x) = \ln f(x), f(x) > 0$

$g'(x) = \dfrac{f'(x)}{f(x)}$

(a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

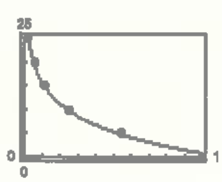
(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2 + 1)$ is not concave up.

93. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$h = 0.8627 - 6.4474 \ln p$.

(c)



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$1 = -6.4474 \dfrac{1}{p}\dfrac{dp}{dh}$ (implicit differentiation)

$\dfrac{dp}{dh} = \dfrac{p}{-6.4474}$

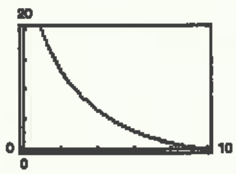
For $h = 5$, $p = 0.55$ and $dp/dh = -0.0853$ atmos/km.

For $h = 20$, $p = 0.06$ and $dp/dh = -0.00931$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

94. $y = 10\ln\left(\dfrac{10+\sqrt{100-x^2}}{x}\right) - \sqrt{100-x^2} = 10\left[\ln\left(10+\sqrt{100-x^2}\right) - \ln x\right] - \sqrt{100-x^2}$

(a)



(b) $$\frac{dy}{dx} = 10\left[\frac{-x}{\sqrt{100-x^2}\left(10+\sqrt{100-x^2}\right)} - \frac{1}{x}\right] + \frac{x}{\sqrt{100-x^2}}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{-10}{10+\sqrt{100-x^2}}\right] - \frac{10}{x} + \frac{x}{\sqrt{100-x^2}}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{-10}{10+\sqrt{100-x^2}} + 1\right] - \frac{10}{x}$$

$$= \frac{x}{\sqrt{100-x^2}}\left[\frac{\sqrt{100-x^2}}{10+\sqrt{100-x^2}}\right] - \frac{10}{x}$$

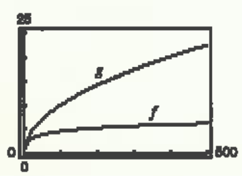
$$= \frac{x}{10+\sqrt{100-x^2}} - \frac{10}{x}$$

$$= \frac{x\left(10-\sqrt{100-x^2}\right)}{x^2} - \frac{10}{x} = -\frac{\sqrt{100-x^2}}{x}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

(c) $\lim_{x\to 10^-} \dfrac{dy}{dx} = 0$

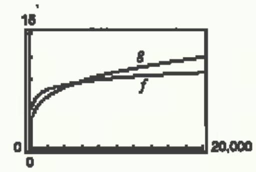
95. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$f'(x) = \dfrac{1}{x}$, $g'(x) = \dfrac{1}{2\sqrt{x}}$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a higher rate than f for "large" values of x.

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$f'(x) = \dfrac{1}{x}$, $g'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a higher rate than f for "large" values of x. $f(x) = \ln x$ increases very slowly for "large" values of x.

96. $y = \ln x$

$y' = \dfrac{1}{x} > 0$ for $x > 0$.

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

97. False

$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$

98. False

π is a constant.

$\dfrac{d}{dx}[\ln \pi] = 0$

Section 5.2 The Natural Logarithmic Function and Integration

1. $u = x + 1,\ du = dx$

$$\int \frac{1}{x+1}\,dx = \ln|x+1| + C$$

2. $u = x - 5,\ du = dx$

$$\int \frac{1}{x-5}\,dx = \ln|x-5| + C$$

3. $u = 3 - 2x,\ du = -2\,dx$

$$\int \frac{1}{3-2x}\,dx = -\frac{1}{2}\int \frac{1}{3-2x}(-2)\,dx = -\frac{1}{2}\ln|3-2x| + C$$

4. $u = 6x + 1,\ du = 6\,dx$

$$\int \frac{1}{6x+1}\,dx = \frac{1}{6}\int \frac{1}{6x+1}(6)\,dx = \frac{1}{6}\ln|6x+1| + C$$

5. $u = x^2 + 1,\ du = 2x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{x^2+1}(2x)\,dx = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C$$

6. $u = 3 - x^3,\ du = -3x^2\,dx$

$$\int \frac{x^2}{3-x^3}\,dx = -\frac{1}{3}\int \frac{1}{3-x^3}(-3x^2)\,dx = -\frac{1}{3}\ln|3-x^3| + C$$

7. $$\int \frac{x^2-4}{x}\,dx = \int\left(x - \frac{4}{x}\right)dx = \frac{x^2}{2} - 4\ln|x| + C$$

8. $u = 9 - x^2,\ du = -2x\,dx$

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = -\frac{1}{2}\int (9-x^2)^{-1/2}(-2x)\,dx = -\sqrt{9-x^2} + C$$

9. $u = x^3 + 3x^2 + 9x,\ du = 3(x^2 + 2x + 3)\,dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx = \frac{1}{3}\int \frac{3(x^2+2x+3)}{x^3+3x^2+9x}\,dx = \frac{1}{3}\ln|x^3+3x^2+9x| + C$$

10. $u = x^2 + 6x + 7,\ du = 2(x+3)\,dx$

$$\int \frac{x+3}{x^2+6x+7}\,dx = \frac{1}{2}\int \frac{2x+6}{x^2+6x+7}\,dx = \frac{1}{2}\ln|x^2+6x+7| + C$$

11. $u = \ln x,\ du = \frac{1}{x}\,dx$

$$\int \frac{(\ln x)^2}{x}\,dx = \frac{1}{3}(\ln x)^3 + C$$

12. $u = \ln x,\ du = \frac{1}{x}\,dx$

$$\int \frac{1}{x\ln x^2}\,dx = \frac{1}{2}\int \frac{1}{x\ln x}\,dx = \frac{1}{2}\ln|\ln|x|| + C$$

13. $u = x + 1,\ du = dx$

$$\int \frac{1}{\sqrt{x+1}}\,dx = \int (x+1)^{-1/2}\,dx = 2(x+1)^{1/2} + C = 2\sqrt{x+1} + C$$

14. $u = 1 + x^{1/3},\ du = \frac{1}{3x^{2/3}}\,dx$

$$\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx = 3\int \frac{1}{1+x^{1/3}}\left(\frac{1}{3x^{2/3}}\right)dx = 3\ln|1+x^{1/3}| + C$$

15. $u = \sqrt{x} - 3,\ du = \dfrac{1}{2\sqrt{x}}\,dx \Rightarrow 2(u + 3)\,du = dx$

$$\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx = 2\int \frac{(u+3)^2}{u}\,du = 2\int \frac{u^2 + 6u + 9}{u}\,du = 2\int \left(u + 6 + \frac{9}{u}\right) du$$
$$= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1 = u^2 + 12u + 18\ln|u| + C_1$$
$$= \left(\sqrt{x} - 3\right)^2 + 12\left(\sqrt{x} - 3\right) + 18\ln\left|\sqrt{x} - 3\right| + C_1$$
$$= x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C \text{ where } C = C_1 - 27.$$

16. $u = 1 + \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx \Rightarrow (u - 1)\,du = dx$

$$\int \frac{1}{1 + \sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int \left(u - \frac{1}{u}\right) du$$
$$= u - \ln|u| + C_1$$
$$= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1$$
$$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

17.
$$\int \frac{2x}{(x-1)^2}\,dx = \int \frac{2x - 2 + 2}{(x-1)^2}\,dx$$
$$= \int \frac{2(x-1)}{(x-1)^2}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$
$$= 2\int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$
$$= 2\ln|x - 1| - \frac{2}{(x-1)} + C$$

18.
$$\int \frac{x(x-2)}{(x-1)^3}\,dx = \int \frac{x^2 - 2x + 1 - 1}{(x-1)^3}\,dx$$
$$= \int \frac{(x-1)^2}{(x-1)^3}\,dx - \int \frac{1}{(x-1)^3}\,dx$$
$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^3}\,dx$$
$$= \ln|x - 1| + \frac{1}{2(x-1)^2} + C$$

19. $\displaystyle\int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|\sin\theta| + C$

$(u = \sin\theta,\ du = \cos\theta\,d\theta)$

20.
$$\int \tan 5\theta\,d\theta = \frac{1}{5}\int \frac{5\sin 5\theta}{\cos 5\theta}\,d\theta$$
$$= -\frac{1}{5}\ln|\cos 5\theta| + C$$

21.
$$\int \csc 2x\,dx = \frac{1}{2}\int (\csc 2x)(2)\,dx$$
$$= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$$

22.
$$\int \sec\frac{x}{2}\,dx = 2\int \sec\frac{x}{2}\left(\frac{1}{2}\right) dx$$
$$= 2\ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right| + C$$

23. $\displaystyle\int \frac{\cos t}{1 + \sin t}\,dt = \ln|1 + \sin t| + C$

24.
$$\int \frac{\sin x}{1 + \cos x}\,dx = -\int \frac{-\sin x}{1 + \cos x}\,dx$$
$$= -\ln|1 + \cos x| + C$$

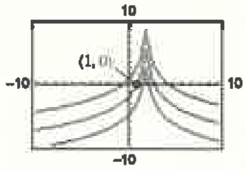
25. $\displaystyle\int \frac{\sec x \tan x}{\sec x - 1}\,dx = \ln|\sec x - 1| + C$

26.
$$\int (\sec t + \tan t)\,dt = \ln|\sec t + \tan t| - \ln|\cos t| + C$$
$$= \ln\left|\frac{\sec t + \tan t}{\cos t}\right| + C = \ln|\sec t(\sec t + \tan t)| + C$$

27. $y = \int \frac{3}{2-x}\,dx$

$= -3\int \frac{1}{x-2}\,dx$

$= -3\ln|x-2| + C$



$(1, 0)$: $0 = -3\ln|1-2| + C \Rightarrow C = 0$

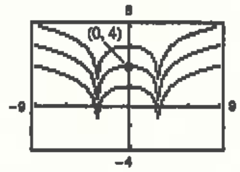
$y = -3\ln|x-2|$

28. $y = \int \frac{2x}{x^2-9}\,dx$

$= \ln|x^2-9| + C$

$(0, 4)$: $4 = \ln|0-9| + C \Rightarrow C = 4 - \ln 9$

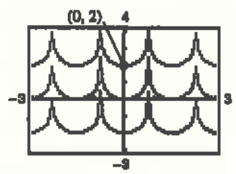
$y = \ln|x^2-9| + 4 - \ln 9$



29. $s = \int \tan(2\theta)\,d\theta$

$= \frac{1}{2}\int \tan(2\theta)(2\,d\theta)$

$= -\frac{1}{2}\ln|\cos 2\theta| + C$



$(0, 2)$: $2 = -\frac{1}{2}\ln|\cos(0)| + C \Rightarrow C = 2$

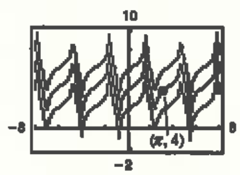
$s = -\frac{1}{2}\ln|\cos 2\theta| + 2$

30. $r = \int \frac{\sec^2 t}{\tan t + 1}\,dt$

$= \ln|\tan t + 1| + C$

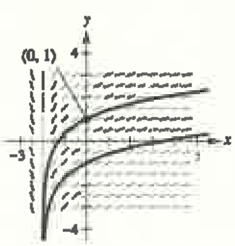
$(\pi, 4)$: $4 = \ln|0 + 1| + C \Rightarrow C = 4$

$r = \ln|\tan t + 1| + 4$



31. $\frac{dy}{dx} = \frac{1}{x+2}$, $(0, 1)$

(a)



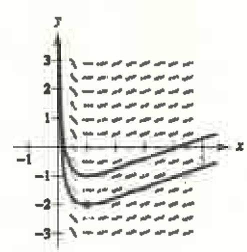
(b) $y = \int \frac{1}{x+2}\,dx = \ln|x+2| + C$

$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$

Hence, $y = \ln|x+2| + 1 - \ln 2 = \ln\left|\frac{x+2}{2}\right| + 1$.

32. $\frac{dy}{dx} = \frac{\ln x}{x}$, $(1, -2)$

(a)



(b) $y = \int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2} + C$

$y(1) = -2 \Rightarrow -2 = \frac{(\ln 1)^2}{2} + C \Rightarrow C = -2$

Hence, $y = \frac{(\ln x)^2}{2} - 2$.

33. $\int_0^4 \frac{5}{3x+1}\,dx = \left[\frac{5}{3}\ln|3x+1|\right]_0^4$

$= \frac{5}{3}\ln 13 \approx 4.275$

34. $\int_{-1}^{1} \frac{1}{x+2}\,dx = \left[\ln|x+2|\right]_{-1}^{1}$

$= \ln 3 - \ln 1 = \ln 3$

35. $u = 1 + \ln x,\ du = \dfrac{1}{x}\,dx$

$$\int_1^e \frac{(1 + \ln x)^2}{x}\,dx = \left[\frac{1}{3}(1 + \ln x)^3\right]_1^e = \frac{7}{3}$$

36. $u = \ln x,\ du = \dfrac{1}{x}\,dx$

$$\int_e^{e^2} \frac{1}{x \ln x}\,dx = \int_e^{e^2} \left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x||\Big]_e^{e^2} = \ln 2$$

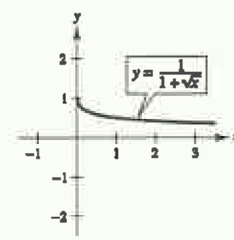
37. $$\int_0^2 \frac{x^2 - 2}{x + 1}\,dx = \int_0^2 \left(x - 1 - \frac{1}{x + 1}\right)dx$$
$$= \left[\frac{1}{2}x^2 - x - \ln|x + 1|\right]_0^2 = -\ln 3$$

38. $$\int_0^1 \frac{x - 1}{x + 1}\,dx = \int_0^1 1\,dx + \int_0^1 \frac{-2}{x + 1}\,dx$$
$$= \Big[x - 2\ln|x + 1|\Big]_0^1 = 1 - 2\ln 2$$

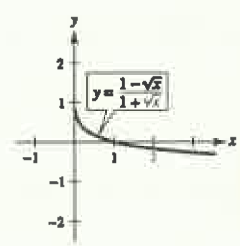
39. $$\int_1^2 \frac{1 - \cos\theta}{\theta - \sin\theta}\,d\theta = \Big[\ln|\theta - \sin\theta|\Big]_1^2 = \ln\left|\frac{2 - \sin 2}{1 - \sin 1}\right| \approx 1.929$$

40. $$\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta = \int_{0.1}^{0.2} (\csc^2 2\theta - 2\csc 2\theta\cot 2\theta + \cot^2 2\theta)\,d\theta$$
$$= \int_{0.1}^{0.2} (2\csc^2 2\theta - 2\csc 2\theta\cot 2\theta - 1)\,d\theta$$
$$= \Big[-\cot 2\theta + \csc 2\theta - \theta\Big]_{0.1}^{0.2} \approx 0.0024$$

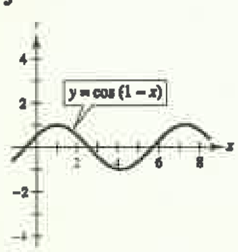
41. $$\int \frac{1}{1 + \sqrt{x}}\,dx = 2\left(1 + \sqrt{x}\right) - 2\ln\left(1 + \sqrt{x}\right) + C_1$$
$$= 2\left[\sqrt{x} - \ln\left(1 + \sqrt{x}\right)\right] + C \text{ where } C = C_1 + 2.$$



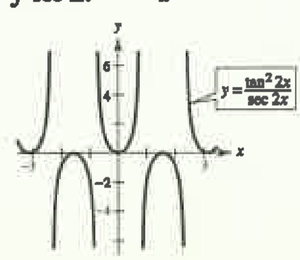
42. $$\int \frac{1 - \sqrt{x}}{1 + \sqrt{x}}\,dx = -\left(1 + \sqrt{x}\right)^2 + 6\left(1 + \sqrt{x}\right) - 4\ln\left(1 + \sqrt{x}\right) + C_1$$
$$= 4\sqrt{x} - x - 4\ln\left(1 + \sqrt{x}\right) + C \text{ where } C = C_1 + 5.$$



43. $$\int \cos(1 - x)\,dx = -\sin(1 - x) + C$$

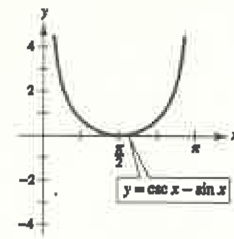


44. $$\int \frac{\tan^2 2x}{\sec 2x}\,dx = \frac{1}{2}\Big[\ln|\sec 2x + \tan 2x| - \sin 2x\Big] + C$$



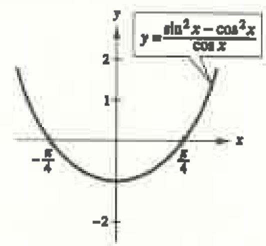
45. $\displaystyle\int_{\pi/4}^{\pi/2}(\csc x - \sin x)\,dx = \Big[-\ln|\csc x + \cot x| + \cos x\Big]_{\pi/4}^{\pi/2}$

$$= \ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{2} \approx 0.174$$



46. $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \Big[\ln|\sec x + \tan x| - 2\sin x\Big]_{-\pi/4}^{\pi/4}$

$$= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2} \approx -1.066$$



47. $-\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$

48. $\ln|\sin x| + C = \ln\left|\frac{1}{\csc x}\right| + C = -\ln|\csc x| + C$

49. $\ln|\sec x + \tan x| + C = \ln\left|\frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C = \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$

$$= \ln\left|\frac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C$$

50. $-\ln|\csc x + \cot x| + C = -\ln\left|\frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C = -\ln\left|\frac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C$

$$= -\ln\left|\frac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C$$

Note: In Exercises 51–54, you can use the Second Fundamental Theorem of Calculus or integrate the function.

51. $F(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$

$F'(x) = \dfrac{1}{x}$

52. $F(x) = \displaystyle\int_0^x \tan t\,dt$

$F'(x) = \tan x$

53. $F(x) = \displaystyle\int_x^{3x}\frac{1}{t}\,dt$

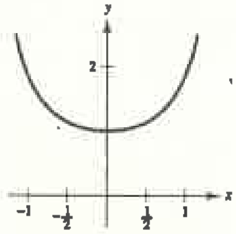
$= \displaystyle\int_1^{3x}\frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt$

$F'(x) = \dfrac{3}{3x} - \dfrac{1}{x} = 0$

54. $F(x) = \displaystyle\int_1^{x^2}\frac{1}{t}\,dt$

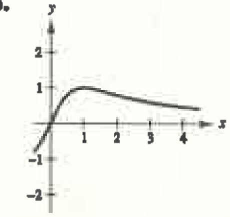
$F'(x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

55.



$A \approx 1.25$
Matches (d)

56.

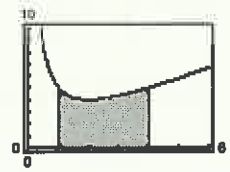


$A \approx 3$
Matches (a)

57. $A = \int_1^4 \frac{x^2+4}{x}\,dx = \int_1^4 \left(x + \frac{4}{x}\right) dx$

$= \left[\frac{x^2}{2} + 4\ln x\right]_1^4 = (8 + 4\ln 4) - \frac{1}{2}$

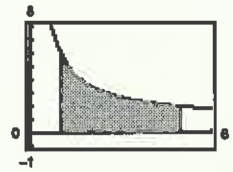
$= \frac{15}{2} + 8\ln 2 \approx 13.045$ square units



58. $A = \int_1^5 \frac{x+5}{x}\,dx = \int_1^5 \left(1 + \frac{5}{x}\right) dx$

$= \left[x + 5\ln x\right]_1^5$

$= 4 + 5\ln 5 \approx 12.047$ square units

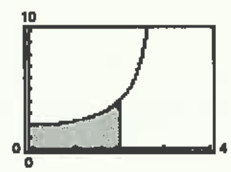


59. $\int_0^2 2\sec\frac{\pi x}{6}\,dx = \frac{12}{\pi}\int_0^2 \sec\left(\frac{\pi x}{6}\right)\frac{\pi}{6}\,dx$

$= \left[\frac{12}{\pi}\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$

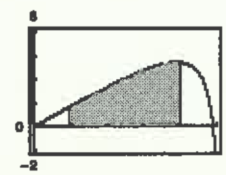
$= \frac{12}{\pi}\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \frac{12}{\pi}\ln|1+0|$

$= \frac{12}{\pi}\ln(2+\sqrt{3}) \approx 5.03041$



60. $\int_1^4 (2x - \tan(0.3x))\,dx = \left[x^2 + \frac{10}{3}\ln|\cos(0.3x)|\right]_1^4$

$= \left[16 + \frac{10}{3}\ln\cos(1.2)\right] - \left[1 + \frac{10}{3}\ln\cos(0.3)\right] \approx 11.7686$



61. $P(t) = \int \frac{3000}{1+0.25t}\,dt = (3000)(4)\int \frac{0.25}{1+0.25t}\,dt = 12{,}000\ln|1+0.25t| + C$

$P(0) = 12{,}000\ln|1+0.25(0)| + C = 1000$

$C = 1000$

$P(t) = 12{,}000\ln|1+0.25t| + 1000 = 1000[12\ln|1+0.25t| + 1]$

$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$

62. $t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T-100}\,dT$

$= \frac{10}{\ln 2}\Big[\ln(T-100)\Big]_{250}^{300} = \frac{10}{\ln 2}[\ln 200 - \ln 150] = \frac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504$ units of time

63. $\frac{1}{50-40}\int_{40}^{50} \frac{90{,}000}{400+3x}\,dx = \Big[3000\ln|400+3x|\Big]_{40}^{50} \approx \$168.27$

64. $\dfrac{dS}{dt} = \dfrac{k}{t}$

$S(t) = \displaystyle\int \frac{k}{t}\,dt = k\ln|t| + C = k\ln t + C$ since $t > 1$.

$S(2) = k\ln 2 + C = 200$

$S(4) = k\ln 4 + C = 300$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

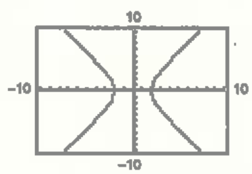
$$S(t) = \frac{100\ln t}{\ln 2} + 100 = 100\left[\frac{\ln t}{\ln 2} + 1\right].$$

65. (a) $2x^2 - y^2 = 8$

$y^2 = 2x^2 - 8$

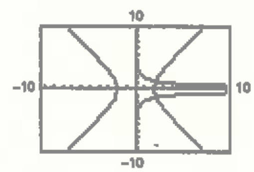
$y_1 = \sqrt{2x^2 - 8}$

$y_2 = -\sqrt{2x^2 - 8}$



(b) $y^2 = e^{-\int(1/x)dx} = e^{-\ln x + C} = e^{\ln(1/x)}(e^C) = \dfrac{1}{x}k$

Let $k = 4$ and graph $y^2 = \dfrac{4}{x}$ $\left(\begin{matrix} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x}\end{matrix}\right)$



(c) In part (a), $2x^2 - y^2 = 8$

$4x - 2yy' = 0$

$y' = \dfrac{2x}{y}.$

In part (b), $y^2 = \dfrac{4}{x} = 4x^{-1}$

$2yy' = \dfrac{-4}{x^2}$

$y' = \dfrac{-2}{yx^2} = \dfrac{-2y}{y^2x^2} = \dfrac{-2y}{4x} = \dfrac{-y}{2x}.$

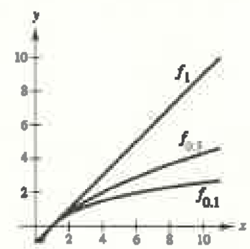
Using a graphing utility the graphs intersect at (2.214, 1.344). The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

66. $k = 1$: $f_1(x) = x - 1$

$k = 0.5$: $f_{0.5}(x) = \dfrac{\sqrt{x} - 1}{0.5} = 2(\sqrt{x} - 1)$

$k = 0.1$: $f_{0.1}(x) = \dfrac{\sqrt[10]{x} - 1}{0.1} = 10(\sqrt[10]{x} - 1)$

$\displaystyle\lim_{k\to 0} f_k(x) = \ln x$



67. False

$\dfrac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$

68. False

$\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$

69. True

$\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C_1$

$= \ln|x| + \ln|C| = \ln|Cx|,\ C \neq 0$

70. False; the integrand has a nonremovable discontinuity at $x = 0$.

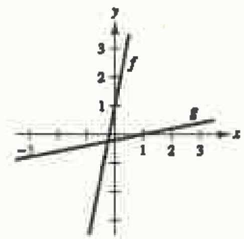
Section 5.3 Inverse Functions

1. (a) $f(x) = 5x + 1$

$g(x) = \dfrac{x-1}{5}$

$f(g(x)) = f\left(\dfrac{x-1}{5}\right) = 5\left(\dfrac{x-1}{5}\right) + 1 = x$

$g((f(x)) = g(5x + 1) = \dfrac{(5x+1) - 1}{5} = x$

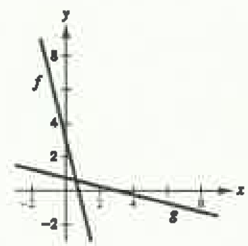
(b) 

2. (a) $f(x) = 3 - 4x$

$g(x) = \dfrac{3-x}{4}$

$f(g(x)) = f\left(\dfrac{3-x}{4}\right) = 3 - 4\left(\dfrac{3-x}{4}\right) = x$

$g(f(x)) = g(3 - 4x) = \dfrac{3 - (3 - 4x)}{4} = x$

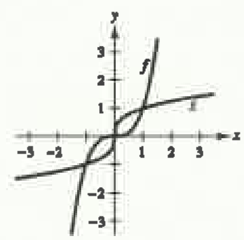
(b) 

3. (a) $f(x) = x^3$

$g(x) = \sqrt[3]{x}$

$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$

$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$

(b) 

4. (a) $f(x) = 1 - x^3$

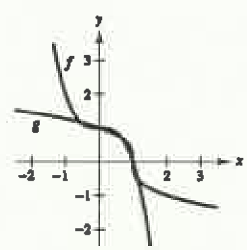
$g(x) = \sqrt[3]{1-x}$

$f(g(x)) = f(\sqrt[3]{1-x}) = 1 - (\sqrt[3]{1-x})^3$

$= 1 - (1 - x) = x$

$g(f(x)) = g(1 - x^3)$

$= \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x$

(b) 

5. (a) $f(x) = \sqrt{x-4}$

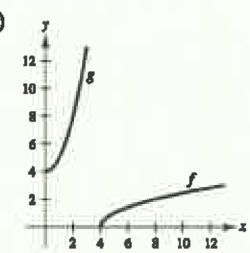
$g(x) = x^2 + 4,\ x \ge 0$

$f(g(x)) = f(x^2 + 4)$

$= \sqrt{(x^2+4) - 4} = \sqrt{x^2} = x$

$g(f(x)) = g(\sqrt{x-4})$

$= (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x$

(b) 

6. (a) $f(x) = 9 - x^2,\ x \ge 0$

$g(x) = \sqrt{9 - x},\ x \le 9$

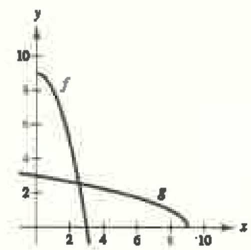
$f(g(x)) = f\left(\sqrt{9 - x}\right)$

$= 9 - \left(\sqrt{9 - x}\right)^2 = 9 - (9 - x) = x$

$g(f(x)) = g(9 - x^2)$

$= \sqrt{9 - (9 - x^2)} = \sqrt{x^2} = x$

(b)



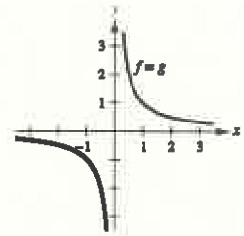
7. (a) $f(x) = \dfrac{1}{x}$

$g(x) = \dfrac{1}{x}$

$f(g(x)) = \dfrac{1}{1/x} = x$

$g(f(x)) = \dfrac{1}{1/x} = x$

(b)



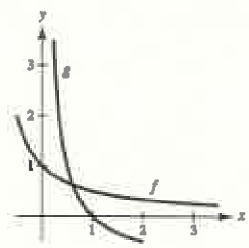
8. (a) $f(x) = \dfrac{1}{1 + x},\ x \ge 0$

$g(x) = \dfrac{1 - x}{x},\ 0 < x \le 1$

$$f(g(x)) = f\left(\frac{1 - x}{x}\right) = \frac{1}{1 + \dfrac{1 - x}{x}} = \frac{1}{\dfrac{1}{x}} = x$$

$$g(f(x)) = g\left(\frac{1}{1 + x}\right) = \frac{1 - \dfrac{1}{1 + x}}{\dfrac{1}{1 + x}} = \frac{x}{1 + x} \cdot \frac{1 + x}{1} = x$$

(b)



9. Matches (c)

10. Matches (b)

11. Matches (a)

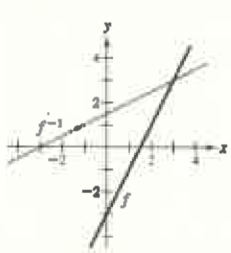
12. Matches (d)

13. $f(x) = 2x - 3 = y$

$x = \dfrac{y + 3}{2}$

$y = \dfrac{x + 3}{2}$

$f^{-1}(x) = \dfrac{x + 3}{2}$

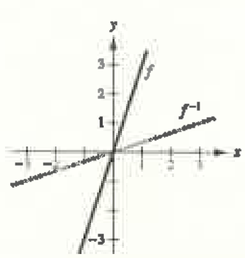


14. $f(x) = 3x = y$

$x = \dfrac{y}{3}$

$y = \dfrac{x}{3}$

$f^{-1}(x) = \dfrac{x}{3}$

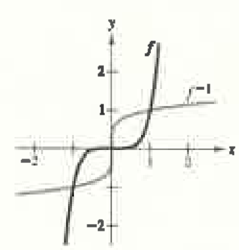


15. $f(x) = x^5 = y$

$x = \sqrt[5]{y}$

$y = \sqrt[5]{x}$

$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$

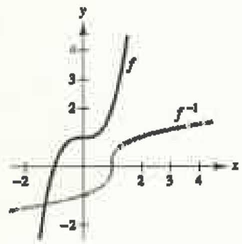


16. $f(x) = x^3 + 1 = y$

$x = \sqrt[3]{y - 1}$

$y = \sqrt[3]{x - 1}$

$f^{-1}(x) = \sqrt[3]{x - 1}$

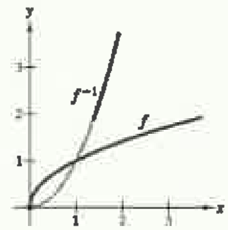


17. $f(x) = \sqrt{x} = y$

$x = y^2$

$y = x^2$

$f^{-1}(x) = x^2, \; x \geq 0$

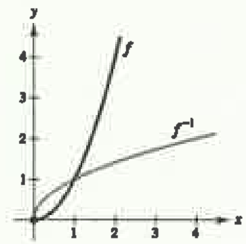


18. $f(x) = x^2 = y, \; 0 \leq x$

$x = \sqrt{y}$

$y = \sqrt{x}$

$f^{-1}(x) = \sqrt{x}$

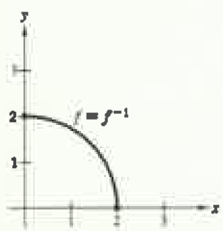


19. $f(x) = \sqrt{4 - x^2} = y, \; 0 \leq x \leq 2$

$x = \sqrt{4 - y^2}$

$y = \sqrt{4 - x^2}$

$f^{-1}(x) = \sqrt{4 - x^2}, \; 0 \leq x \leq 2$

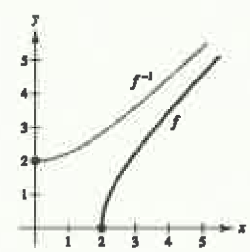


20. $f(x) = \sqrt{x^2 - 4} = y, \; x \geq 2$

$x = \sqrt{y^2 + 4}$

$y = \sqrt{x^2 + 4}$

$f^{-1}(x) = \sqrt{x^2 + 4}, \; x \geq 0$

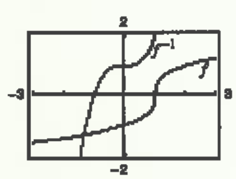


21. $f(x) = \sqrt[3]{x - 1} = y$

$x = y^3 + 1$

$y = x^3 + 1$

$f^{-1}(x) = x^3 + 1$



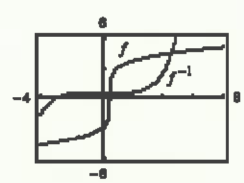
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

22. $f(x) = 3\sqrt[5]{2x - 1} = y$

$$x = \frac{y^5 + 243}{486}$$

$$y = \frac{x^5 + 243}{486}$$

$$f^{-1}(x) = \frac{x^5 + 243}{486}$$



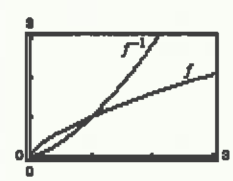
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

23. $f(x) = x^{2/3} = y, \; x \geq 0$

$x = y^{3/2}$

$y = x^{3/2}$

$f^{-1}(x) = x^{3/2}, \; x \geq 0$



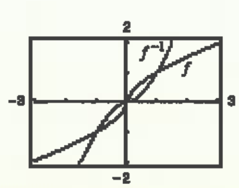
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

24. $f(x) = x^{3/5} = y$

$x = y^{5/3}$

$y = x^{5/3}$

$f^{-1}(x) = x^{5/3}$



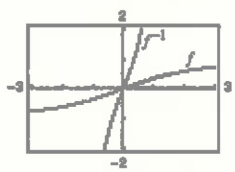
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

25. $f(x) = \dfrac{x}{\sqrt{x^2+7}} = y$

$$x = \frac{\sqrt{7}y}{\sqrt{1-y^2}}$$

$$y = \frac{\sqrt{7}x}{\sqrt{1-x^2}}$$

$$f^{-1}(x) = \frac{\sqrt{7}x}{\sqrt{1-x^2}}, \ -1 < x < 1$$



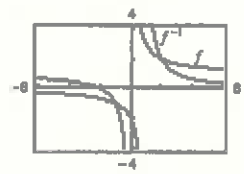
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

26. $f(x) = \dfrac{x+2}{x} = y$

$$x = \frac{2}{y-1}$$

$$y = \frac{2}{x-1}$$

$$f^{-1}(x) = \frac{2}{x-1}$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

27. $f(x) = \dfrac{x}{x^2-4} = y$ on $(-2, 2)$

$$x^2y - 4y = x$$

$$x^2y - x - 4y = 0$$

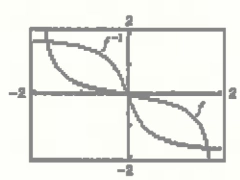
$$a = y, b = -1, c = -4y$$

$$x = \frac{1 \pm \sqrt{1-4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1+16y^2}}{2y}$$

$$y = f^{-1}(x) = \begin{cases} (1-\sqrt{1+16x^2})/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Domain: all x

Range: $-2 < y < 2$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

28. $f(x) = 2 - \dfrac{3}{x^2} = y$ on $(0, 10)$

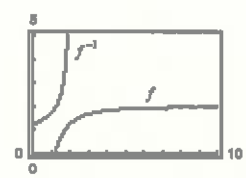
$$2x^2 - 3 = x^2y$$

$$x^2(2-y) = 3$$

$$x = \pm\sqrt{\frac{3}{2-y}}$$

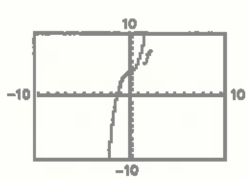
$$y = \pm\sqrt{\frac{3}{2-x}}$$

$$f^{-1}(x) = \sqrt{\frac{3}{2-x}}, \ x < 2$$

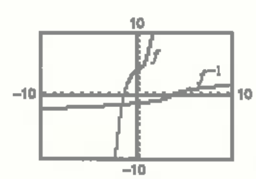


The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

29. (a)

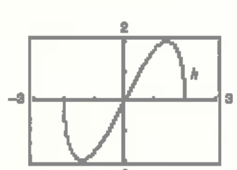


(b)

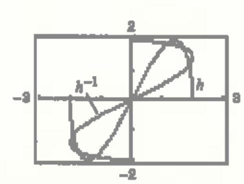


(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

30. (a)



(b)



(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

49. $f(x) = \dfrac{x^4}{4} - 2x^2$

$f'(x) = x^3 - 4x = 0$ when $x = 0, 2, -2$.

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

50. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2 \geq 0$ for all x.

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = 2 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$ for all x.

f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

52. $f(x) = \ln(x-3), x > 3$

$f'(x) = \dfrac{1}{x-3} > 0$ for $x > 3$.

f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = (x-4)^2$ on $[4, \infty)$

$f'(x) = 2(x-4) > 0$ on $(4, \infty)$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

54. $f(x) = |x+2|$ on $[-2, \infty)$

$f'(x) = \dfrac{|x+2|}{x+2}(1) = 1 > 0$ on $(-2, \infty)$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

55. $f(x) = \dfrac{4}{x^2}$ on $(0, \infty)$

$f'(x) = -\dfrac{8}{x^3} < 0$ on $(0, \infty)$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

56. $f(x) = \tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$f'(x) = \sec^2 x > 0$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

f is increasing on $(-\pi/2, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

57. $f(x) = \cos x$ on $[0, \pi]$

$f'(x) = -\sin x < 0$ on $(0, \pi)$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

58. $f(x) = \sec x$ on $\left[0, \dfrac{\pi}{2}\right)$

$f'(x) = \sec x \tan x > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

f is increasing on $[0, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

59. f is not one-to-one because many different x-values yield the same y-value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\dfrac{(2n-1)\pi}{2}$

60. f is not one-to-one because different x-values yield the same y-value.

Example: $f(3) = f\left(-\dfrac{4}{3}\right) = \dfrac{3}{5}$

Not continuous at ± 2.

61. $f(x) = \sqrt{x-2}$, Domain: $x \geq 2$

$f'(x) = \dfrac{1}{2\sqrt{x-2}} > 0$ for $x > 2$.

f is one-to-one; has an inverse

$$\sqrt{x-2} = y$$
$$x - 2 = y^2$$
$$x = y^2 + 2$$
$$y = x^2 + 2$$
$$f^{-1}(x) = x^2 + 2, x \geq 0$$

62. $f(x) = -3$

Not one-to-one; does not have an inverse

63. $f(x) = |x - 2|, x \le 2$

$= -(x - 2)$

$= 2 - x$

f is one-to-one; has an inverse

$2 - x = y$

$2 - y = x$

$f^{-1}(x) = 2 - x,\ x \ge 0$

64. $f(x) = ax + b$

f is one-to-one; has an inverse

$ax + b = y$

$x = \dfrac{y - b}{a}$

$y = \dfrac{x - b}{a}$

$f^{-1}(x) = \dfrac{x - b}{a},\ a \ne 0$

65. $f(x) = (x - 3)^2$ is one-to-one for $x \ge 3$.

$(x - 3)^2 = y$

$x - 3 = \sqrt{y}$

$x = \sqrt{y} + 3$

$y = \sqrt{x} + 3$

$f^{-1}(x) = \sqrt{x} + 3,\ x \ge 0$

66. $f(x) = 16 - x^4$ is one-to-one for $x \ge 0$.

$16 - x^4 = y$

$16 - y = x^4$

$\sqrt[4]{16 - y} = x$

$\sqrt[4]{16 - x} = y$

$f^{-1}(x) = \sqrt[4]{16 - x},\ x \le 16$

67. $f(x) = |x + 3|$ is one-to-one for $x \ge -3$.

$x + 3 = y$

$x = y - 3$

$y = x - 3$

$f^{-1}(x) = x - 3,\ x \ge 0$

68. $f(x) = |x - 3|$ is one-to-one for $x \ge 3$.

$x - 3 = y$

$x = y + 3$

$y = x + 3$

$f^{-1}(x) = x + 3,\ x \ge 0$

69. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V.

70. No, there could be two times $t_1 \ne t_2$ for which $h(t_1) = h(t_2)$.

71. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

72. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A.

73. $f(x) = x^3 + 2x - 1,\ f(1) = 2 = a$

$f'(x) = 3x^2 + 2$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

74. $f(x) = 2x^5 + x^3 + 1, f(-1) = -2 = a$

$f'(x) = 10x^4 + 3x^2$

$$(f^{-1})'(-2) = \frac{1}{f'(f^{-1}(-2))} = \frac{1}{f'(-1)}$$

$$= \frac{1}{10(-1)^4 + 3(-1)^2} = \frac{1}{13}$$

75. $f(x) = \sin x, f\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2} = a$

$f'(x) = \cos x$

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)}$$

$$= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

76. $f(x) = \cos 2x, f(0) = 1 = a$

$f'(x) = -2\sin 2x$

$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{-2\sin 0} = \dfrac{1}{0}$ which is undefined.

77. $f(x) = x^3 - \dfrac{4}{x}, f(2) = 6 = a$

$f'(x) = 3x^2 + \dfrac{4}{x^2}$

$(f^{-1})'(6) = \dfrac{1}{f'(f^{-1}(6))} = \dfrac{1}{f'(2)} = \dfrac{1}{3(2)^2 + (4/2^2)} = \dfrac{1}{13}$

78. $f(x) = \sqrt{x-4}, f(8) = 2 = a$

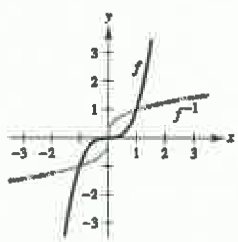
$f'(x) = \dfrac{1}{2\sqrt{x-4}}$

$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(8)} = \dfrac{1}{1/(2\sqrt{8-4})} = \dfrac{1}{1/4} = 4$

79. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = x^3, \left(\dfrac{1}{2}, \dfrac{1}{8}\right)$

$f'(x) = 3x^2$

$f'\left(\dfrac{1}{2}\right) = \dfrac{3}{4}$

$f^{-1}(x) = \sqrt[3]{x}, \left(\dfrac{1}{8}, \dfrac{1}{2}\right)$

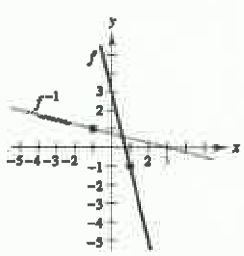
$(f^{-1})'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$

$(f^{-1})'\left(\dfrac{1}{8}\right) = \dfrac{4}{3}$

80. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = 3 - 4x, (1, -1)$

$f'(x) = -4$

$f'(1) = -4$

$f^{-1}(x) = \dfrac{3-x}{4}, (-1, 1)$

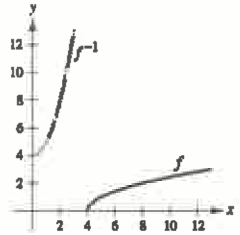
$(f^{-1})'(x) = -\dfrac{1}{4}$

$(f^{-1})'(-1) = -\dfrac{1}{4}$

81. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)



(d) $f(x) = \sqrt{x - 4}$, $(5, 1)$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$f'(5) = \frac{1}{2}$$

$$f^{-1}(x) = x^2 + 4, \ (1, 5)$$

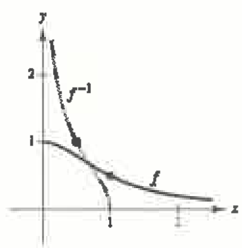
$$(f^{-1})'(x) = 2x$$

$$(f^{-1})'(1) = 2$$

82. (a) Domain $f = [0, \infty)$, Domain $f^{-1}(0, 1]$

(b) Range $f = (0, 1]$, Range $f^{-1} = [0, \infty)$

(c)



(d) $f(x) = \dfrac{1}{1 + x^2}$, $x \ge 0$, $\left(1, \dfrac{1}{2}\right)$

$$f'(x) = \frac{-2x}{(1 + x^2)^2}$$

$$f'(1) = -\frac{1}{2}$$

$$f^{-1}(x) = \sqrt{\frac{1-x}{x}}, \ \left(\frac{1}{2}, 1\right)$$

$$(f^{-1})'(x) = -\frac{1}{2x^2}\sqrt{\frac{x}{1-x}}$$

$$(f^{-1})'\left(\frac{1}{2}\right) = -2$$

83. $x = y^3 - 7y^2 + 2$

$$1 = 3y^2\frac{dy}{dx} - 14y\frac{dy}{dx}$$

$\dfrac{dy}{dx} = \dfrac{1}{3y^2 - 14y}$. At $(-4, 1)$, $\dfrac{dy}{dx} = \dfrac{1}{3 - 14} = \dfrac{-1}{11}$.

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.

Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

Hence, $\dfrac{dy}{dx} = \dfrac{1}{-11} = \dfrac{-1}{11}$.

84. $x = 2\ln(y^2 - 3)$

$$1 = 2\frac{1}{y^2 - 3}2y\frac{dy}{dx}$$

$\dfrac{dy}{dx} = \dfrac{y^2 - 3}{4y}$. At $(0, 4)$, $\dfrac{dy}{dx} = \dfrac{16 - 3}{16} = \dfrac{13}{16}$.

In Exercises 85–88, use the following.

$$f(x) = \tfrac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x + 3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

85. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

86. $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3) = g^{-1}(0) = 0$

87. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

88. $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}\left(\sqrt[3]{-4}\right)$
$= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[9]{4}$

In Exercises 89–92, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x+5}{2}$$

89. $(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$

$$= g^{-1}(x - 4)$$
$$= \frac{(x-4)+5}{2}$$
$$= \frac{x+1}{2}$$

90. $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$

$$= f^{-1}\left(\frac{x+5}{2}\right)$$
$$= \frac{x+5}{2} - 4$$
$$= \frac{x-3}{2}$$

91. $(f \circ g)(x) = f(g(x))$

$$= f(2x - 5)$$
$$= (2x - 5) + 4$$
$$= 2x - 1$$

Hence, $(f \circ g)^{-1}(x) = \dfrac{x+1}{2}$

(Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$)

92. $(g \circ f)(x) = g(f(x))$

$$= g(x + 4)$$
$$= 2(x + 4) - 5$$
$$= 2x + 3$$

Hence, $(g \circ f)^{-1}(x) = \dfrac{x-3}{2}$

(Note: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$)

93. Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$$(f \circ g)(x) = y$$
$$f(g(x)) = y$$
$$g(x) = f^{-1}(y)$$
$$x = g^{-1}(f^{-1}(y))$$
$$= (g^{-1} \circ f^{-1})(y)$$

Since f and g are one-to-one functions, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

94. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

95. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a). Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

96. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \Rightarrow x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

97. False

Let $f(x) = x^2$.

98. True; if f has a y-intercept.

99. True

100. False

Let $f(x) = x$ or $g(x) = 1/x$.

101. Not true

$$\text{Let } f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1 - x, & 1 < x \le 2 \end{cases}.$$

f is one-to-one, but not strictly monotonic.

102. From Theorem 5.9, we have:

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x)) \cdot [1/(f'(g(x)))]}{[f'(g(x))]^2}$$

$$= -\frac{f''(g(x))}{[f'(g(x))]^3}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

103. $$f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}},\ f(2) = 0$$

$$f'(x) = \frac{1}{\sqrt{1+x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^0 = 1$

$\ln 1 = 0$

2. $e^{-2} = 0.1353\ldots$

$\ln 0.1353\ldots = -2$

3. $\ln 2 = 0.6931\ldots$

$e^{0.6931\ldots} = 2$

4. $\ln 0.5 = -0.6931\ldots$

$e^{-0.6931\ldots} = \frac{1}{2}$

5. $e^{\ln x} = 4$

$x = 4$

6. $e^{\ln 2x} = 12$

$2x = 12$

$x = 6$

7. $\ln x = 2$

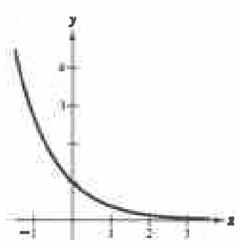
$x = e^2 \approx 7.3891$

8. $\ln x^2 = 10$

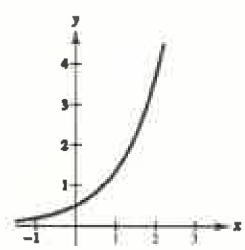
$x^2 = e^{10}$

$x = \pm e^5 \approx \pm 148.4132$

9. $y = e^{-x}$



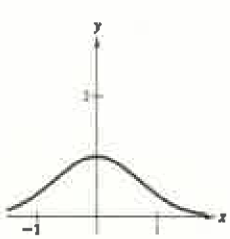
10. $y = \frac{1}{2}e^x$



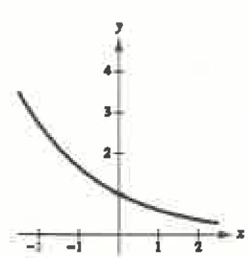
11. $y = e^{-x^2}$

Symmetric with respect to the y-axis

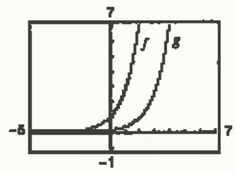
Horizontal asymptote: $y = 0$



12. $y = e^{-x/2}$

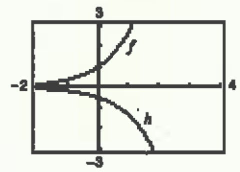


13. (a)



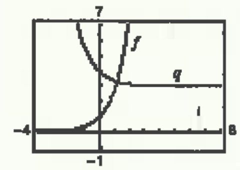
Horizontal shift 2 units to the right

(b)



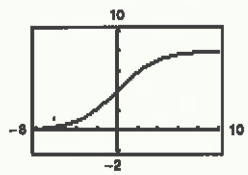
A reflection in the x-axis and a vertical shrink

(c)



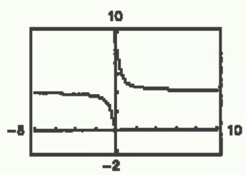
Vertical shift 3 units upward and a reflection in the y-axis

14. (a)



Horizontal asymptotes: $y = 0$ and $y = 8$

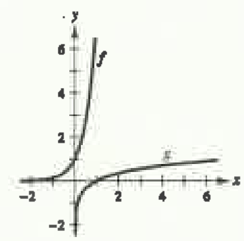
(b)



Horizontal asymptote: $y = 4$

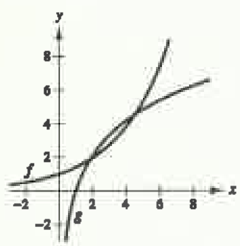
15. $f(x) = e^{2x}$

$g(x) = \ln\sqrt{x} = \dfrac{1}{2}\ln x$



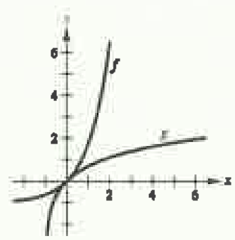
16. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3\ln x$



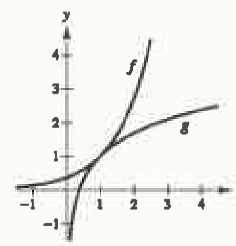
17. $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$



18. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



19. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

20. $y = Ce^{-ax}$

Horizontal asymptote: $y = 0$

Reflection in the y-axis

Matches (d)

21. $y = C(1 - e^{-ax})$

Vertical shift C units

Reflection in both the x- and y-axes

Matches (a)

22. $y = \dfrac{C}{1 + e^{-ax}}$

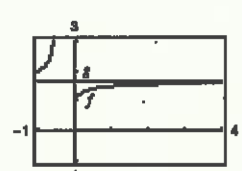
$$\lim_{x\to\infty} \frac{C}{1 + e^{-ax}} = C$$

$$\lim_{x\to-\infty} \frac{C}{1 + e^{-ax}} = 0$$

Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

23.



As $x \to \infty$, the graph of f approaches the graph of g.

$$\lim_{x\to\infty}\left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

24. In the same way,

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

25. $\left(1+\dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000} \approx 2.718280469$

$e \approx 2.718281828$

$e > \left(1+\dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

26. $1+1+\dfrac{1}{2}+\dfrac{1}{6}+\dfrac{1}{24}+\dfrac{1}{120}+\dfrac{1}{720}+\dfrac{1}{5040} = 2.71\overline{825396}$

$e \approx 2.718281828$

$e > 1+1+\dfrac{1}{2}+\dfrac{1}{6}+\dfrac{1}{24}+\dfrac{1}{120}+\dfrac{1}{720}+\dfrac{1}{5040}$

27. (a) $y = e^{3x}$

$y' = 3e^{3x}$

At $(0, 1)$, $y' = 3$.

(b) $y = e^{-3x}$

$y' = -3e^{-3x}$

At $(0, 1)$, $y' = -3$.

28. (a) $y = e^{2x}$

$y' = 2e^{2x}$

At $(0, 1)$, $y' = 2$.

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

At $(0, 1)$, $y' = -2$.

29. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

30. $f(x) = e^{1-x}$

$f'(x) = -e^{1-x}$

31. $f(x) = e^{-2x+x^2}$

$\dfrac{dy}{dx} = 2(x-1)e^{-2x+x^2}$

32. $y = e^{-x^2}$

$\dfrac{dy}{dx} = -2xe^{-x^2}$

33. $y = e^{\sqrt{x}}$

$\dfrac{dy}{dx} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

34. $y = x^2e^{-x}$

$\dfrac{dy}{dx} = -x^2e^{-x} + 2xe^{-x}$

$= xe^{-x}(2-x)$

35. $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$

36. $g(t) = e^{-1/t^2}$

$g'(t) = \dfrac{2e^{-1/t^2}}{t^3}$

37. $y = \ln e^{x^2} = x^2$

$\dfrac{dy}{dx} = 2x$

38. $y = \ln\left(\dfrac{1+e^x}{1-e^x}\right)$

$= \ln(1+e^x) - \ln(1-e^x)$

$\dfrac{dy}{dx} = \dfrac{e^x}{1+e^x} + \dfrac{e^x}{1-e^x}$

$= \dfrac{2e^x}{1-e^{2x}}$

39. $y = \ln(1+e^{2x})$

$\dfrac{dy}{dx} = \dfrac{2e^{2x}}{1+e^{2x}}$

40. $y = \ln\left(\dfrac{e^x+e^{-x}}{2}\right)$

$= \ln(e^x+e^{-x}) - \ln 2$

$\dfrac{dy}{dx} = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

$= \dfrac{e^{2x}-1}{e^{2x}+1}$

41. $y = \dfrac{2}{e^x+e^{-x}} = 2(e^x+e^{-x})^{-1}$

$\dfrac{dy}{dx} = -2(e^x+e^{-x})^{-2}(e^x-e^{-x})$

$= \dfrac{-2(e^x-e^{-x})}{(e^x+e^{-x})^2}$

42. $y = \dfrac{e^x-e^{-x}}{2}$

$\dfrac{dy}{dx} = \dfrac{e^x+e^{-x}}{2}$

43. $y = x^2e^x - 2xe^x + 2e^x = e^x(x^2 - 2x + 2)$

$$\frac{dy}{dx} = e^x(2x - 2) + e^x(x^2 - 2x + 2) = x^2e^x$$

44. $y = xe^x - e^x = e^x(x - 1)$

$$\frac{dy}{dx} = e^x + e^x(x - 1) = xe^x$$

45. $f(x) = e^{-x}\ln x$

$$f'(x) = e^{-x}\left(\frac{1}{x}\right) - e^{-x}\ln x = e^{-x}\left(\frac{1}{x} - \ln x\right)$$

46. $f(x) = e^3\ln x$

$$f'(x) = \frac{e^3}{x}$$

47. $y = e^x(\sin x + \cos x)$

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)(e^x)$$

$$= e^x(2\cos x) = 2e^x\cos x$$

48. $y = \ln e^x = x$

$$\frac{dy}{dx} = 1$$

49. $xe^y - 10x + 3y = 0$

$$xe^y\frac{dy}{dx} + e^y - 10 + 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^y + 3) = 10 - e^y$$

$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

50. $e^{xy} + x^2 - y^2 = 10$

$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

51. $f(x) = (3 + 2x)e^{-3x}$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$$

$$= (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$$

$$= 3(6x + 5)e^{-3x}$$

52. $g(x) = \sqrt{x} + e^x\ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x\ln x$$

$$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x\ln x$$

$$= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x\ln x$$

53.

$$y = e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$$

$$y' = e^x\left(-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x\right) + e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$$

$$= e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$$

$$y'' = e^x\left[-\left(\sqrt{2} + 2\right)\sin\sqrt{2}x + \left(\sqrt{2} - 2\right)\cos\sqrt{2}x\right] + e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$$

$$= e^x\left[\left(-1 - 2\sqrt{2}\right)\sin\sqrt{2}x + \left(-1 + 2\sqrt{2}\right)\cos\sqrt{2}x\right]$$

$$-2y' + 3y = -2e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right] + 3e^x\left[\cos\sqrt{2}x + \sin\sqrt{2}x\right]$$

$$= e^x\left[\left(1 - 2\sqrt{2}\right)\cos\sqrt{2}x + \left(1 + 2\sqrt{2}\right)\sin\sqrt{2}x\right] = -y''$$

Therefore, $-2y' + 3y = -y'' \Rightarrow y'' - 2y' + 3y = 0$.

54. $y = e^x(3\cos 2x - 4\sin 2x)$

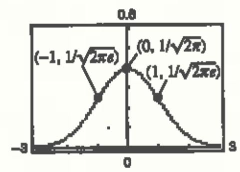
$$y' = e^x(-6\sin 2x - 8\cos 2x) + e^x(3\cos 2x - 4\sin 2x)$$
$$= e^x(-10\sin 2x - 5\cos 2x) = -5e^x(2\sin 2x + \cos 2x)$$

$$y'' = -5e^x(4\cos 2x - 2\sin 2x) - 5e^x(2\sin 2x + \cos 2x) = -5e^x(5\cos 2x) = -25e^x\cos 2x$$

$$y'' - 2y' = -25e^x\cos 2x - 2(-5e^x)(2\sin 2x + \cos 2x) = -5e^x(3\cos 2x - 4\sin 2x) = -5y$$

Therefore, $y'' - 2y' = -5y \Rightarrow y'' - 2y' + 5y = 0.$

55. $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$



$f'(x) = \dfrac{1}{\sqrt{2\pi}}(-xe^{-x^2/2}) = 0$ when $x = 0$.

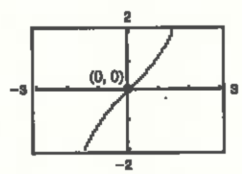
$f''(x) = \dfrac{1}{\sqrt{2\pi}}(x^2e^{-x^2/2} - e^{-x^2/2})$

$= \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}(x^2 - 1) = 0$ when $x = \pm 1$.

Relative maximum: $\left(0, \dfrac{1}{\sqrt{2\pi}}\right)$

Points of inflection: $\left(\pm 1, \dfrac{1}{\sqrt{2e\pi}}\right)$

56. $f(x) = \dfrac{e^x - e^{-x}}{2}$

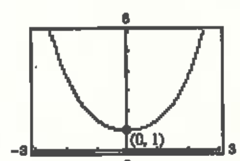


$f'(x) = \dfrac{e^x + e^{-x}}{2} > 0$

$f''(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

Point of inflection: $(0, 0)$

57. $f(x) = \dfrac{e^x + e^{-x}}{2}$

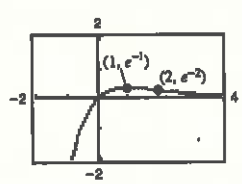


$f'(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

$f''(x) = \dfrac{e^x + e^{-x}}{2} > 0$

Relative minimum: $(0, 1)$

58. $f(x) = xe^{-x}$



$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x) = 0$ when $x = 1$.

$f''(x) = -e^{-x} + (-e^{-x})(1 - x) = e^{-x}(x - 2) = 0$ when $x = 2$.

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$

59. $f(x) = x^2e^{-x}$

$f'(x) = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2 - x) = 0$ when $x = 0, 2$.

$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$

$= e^{-x}(x^2 - 4x + 2) = 0$ when $x = 2 \pm \sqrt{2}$.

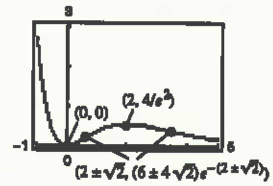
Relative minimum: $(0, 0)$

Relative maximum: $(2, 4e^{-2})$

$x = 2 \pm \sqrt{2}$

$y = (2 \pm \sqrt{2})^2e^{-(2\pm\sqrt{2})}$

Points of inflection: $(3.414, 0.384)$, $(0.586, 0.191)$



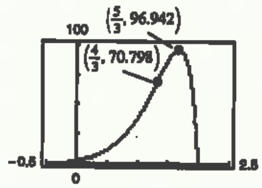
60. $f(x) = -2 + e^{3x}(4 - 2x)$

$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0$ when $x = \frac{5}{3}$.

$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0$ when $x = \frac{4}{3}$.

Relative maximum: $\left(\frac{5}{3}, 96.942\right)$

Point of inflection: $\left(\frac{4}{3}, 70.798\right)$

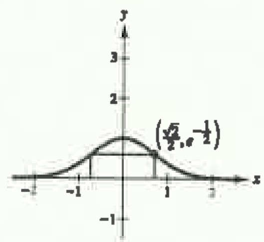


61. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$

$= 2e^{-x^2}(1 - 2x^2) = 0$ when $x = \frac{\sqrt{2}}{2}$.

$A = \sqrt{2}e^{-1/2}$



62. (a) $f(c) = f(c + x)$

$10ce^{-c} = 10(c + x)e^{-(c+x)}$

$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$

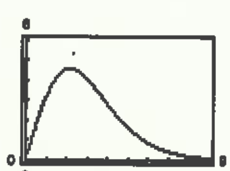
$ce^{c+x} = (c + x)e^c$

$ce^x = c + x$

$ce^x - c = x$

$c = \frac{x}{e^x - 1}$

(c) $A(x) = \frac{10x^2}{e^x - 1}e^{x/(1-e^x)}$

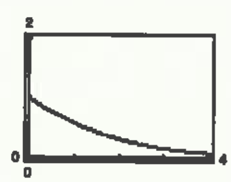


The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(b) $A(x) = xf(c) = x\left[10\left(\frac{x}{e^x - 1}\right)e^{-(x/(e^x-1))}\right]$

$= \frac{10x^2}{e^x - 1}e^{x/(1-e^x)}$

(d) $c = \frac{x}{e^x - 1}$



$\lim_{x\to 0^+} c = 1$

$\lim_{x\to\infty} c = 0$

63. $y = \dfrac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2(1 + ae^{-x/b})\left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

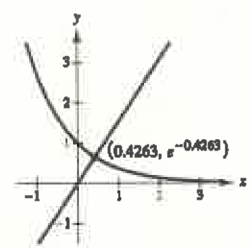
$$= \frac{Lae^{-x/b}[ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$y'' = 0$ if $ae^{-x/b} = 1 \Rightarrow \dfrac{-x}{b} = \ln\left(\dfrac{1}{a}\right) \Rightarrow x = b \ln a$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

64. Let (x_0, y_0) be the desired point on $y = e^{-x}$.



$y = e^{-x}$

$y' = -e^{-x}$ (Slope of tangent line)

$-\dfrac{1}{y'} = e^{x}$ (Slope of normal line)

$y - e^{-x_0} = e^{x_0}(x - x_0)$

We want $(0, 0)$ to satisfy the equation:

$$-e^{-x_0} = -x_0e^{x_0}$$

$$1 = x_0e^{2x_0}$$

$$x_0e^{2x_0} - 1 = 0$$

Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$(0.4263, e^{-0.4263})$

65. $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

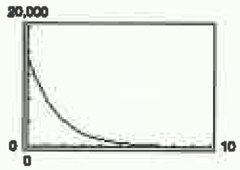
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

We approximate the root of f to be $x = 0.567$.

66. $V = 15{,}000e^{-0.6286t}, 0 \le t \le 10$

(a)

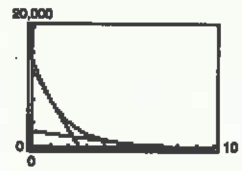


(b) $\frac{dV}{dt} = -9429e^{-0.6286t}$

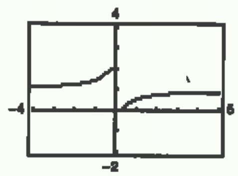
When $t = 1, \frac{dV}{dt} \approx -5028.84$.

When $t = 5, \frac{dV}{dt} \approx -406.89$.

(c)

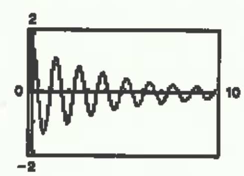


67. (a)



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

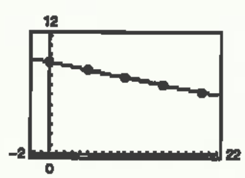
68. $1.56e^{-0.22t} \cos 4.9t \le 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \ge 7.79$ seconds.



69.

h	0	5	10	15	20
P	10,332	5,583	2,376	1,240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248

(a)



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.

(b) $\ln P = ah + b$

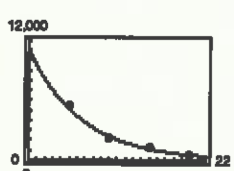
$P = e^{ah+b} = e^b e^{ah}$

$P = Ce^{ah}, C = e^b$

For our data, $a = -0.1499$ and $C = e^{9.3018} = 10{,}957.7$

$P = 10{,}957.7e^{-0.1499h}$

(c)



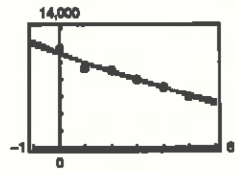
(d) $\frac{dP}{dh} = (10{,}957.71)(-0.1499)e^{-0.1499h}$

$= -1642.56e^{-0.1499h}$

For $h = 5, \frac{dP}{dh} = -776.3$. For $h = 18, \frac{dP}{dh} \approx -110.6$.

70. (a) $y = -995.57t + 11{,}018.10$

$y = 53.39t^2 - 1262.54t + 11{,}196.07$



(b) The slope represents the rate of decrease in value of the car.

(c) $\ln V = -0.1178t + 9.325$

$V = e^{-0.1178t+9.325} = 11{,}215.5e^{-0.1178t}$

(d) Horizontal asymptote: $V = 0$

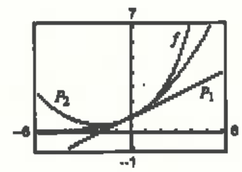
As $t \to \infty$, the value of the car approaches 0.

(e) $\dfrac{dV}{dt} = -1321.2e^{-0.1178t}$

For $t = 1$, $\dfrac{dV}{dt} \approx -1174.38$ dollars/yr.

For $t = 5$, $\dfrac{dV}{dt} \approx -733.11$ dollars/yr.

71. $f(x) = e^{x/2}, f(0) = 1$



$f'(x) = \frac{1}{2}e^{x/2}, f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{4}e^{x/2}, f''(0) = \frac{1}{4}$

$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, P_1(0) = 1$

$P_1'(x) = \frac{1}{2}, P_1'(0) = \frac{1}{2}$

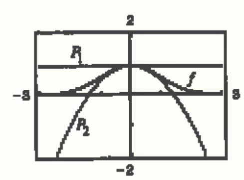
$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 = \frac{x^2}{8} + \frac{x}{2} + 1, P_2(0) = 1$

$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, P_2'(0) = \frac{1}{2}$

$P_2''(x) = \frac{1}{4}, P_2''(0) = \frac{1}{4}$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

72. $f(x) = e^{-x^2/2}, f(0) = 1$



$f'(x) = -xe^{-x^2/2}, f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1), f''(0) = -1$

$P_1(x) = 1 + 0(x - 0) = 1, P_1(0) = 1$

$P_1'(x) = 0, P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \frac{1}{2}(x - 0)^2 = 1 - \frac{x^2}{2}, P_2(0) = 1$

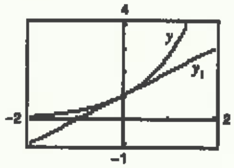
$P_2'(x) = -x, P_2'(0) = 0$

$P_2''(x) = -1, P_2''(0) = -1$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

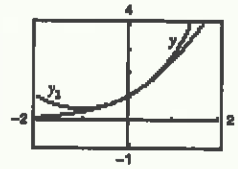
73. (a) $y = e^x$

$y_1 = 1 + x$



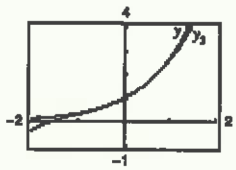
(b) $y = e^x$

$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$



(c) $y = e^x$

$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$



74. n^{th} term is $x^n/n!$ in polynomial:

$$y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Conjecture: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

75. Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}\,5dx = e^{5x} + C$$

76. Let $u = -x^4$, $du = -4x^3\,dx$.

$$\int e^{-x^4}(-4x^3)\,dx = e^{-x^4} + C$$

77. Let $u = -2x$, $du = -2\,dx$.

$$\int_0^1 e^{-2x}dx = -\frac{1}{2}\int_0^1 e^{-2x}(-2)\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1 = \frac{1}{2}(1 - e^{-2}) = \frac{e^2 - 1}{2e^2}$$

78. Let $u = 1 - x$, $du = -dx$.

$$\int_1^2 e^{1-x}dx = -\int_1^2 e^{1-x}(-1)\,dx = \left[-e^{1-x}\right]_1^2 = 1 - e^{-1} = \frac{e - 1}{e}$$

79. Let $u = 1 + e^{-x}$, $du = -e^{-x}dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}}dx = -\int \frac{-e^{-x}}{1 + e^{-x}}dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

80. Let $u = 1 + e^{2x}$, $du = 2e^{2x}dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}}dx = \frac{1}{2}\int \frac{2e^{2x}}{1 + e^{2x}}dx = \frac{1}{2}\ln(1 + e^{2x}) + C$$

81. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2}dx$.

$$\int_1^3 \frac{e^{3/x}}{x^2}dx = -\frac{1}{3}\int_1^3 e^{3/x}\left(-\frac{3}{x^2}\right)dx = \left[-\frac{1}{3}e^{3/x}\right]_1^3 = \frac{e}{3}(e^2 - 1)$$

82. Let $u = \frac{-x^2}{2}$, $du = -x\,dx$.

$$\int_0^{\sqrt{2}} xe^{-x^2/2}\,dx = -\int_0^{\sqrt{2}} e^{-x^2/2}(-x)\,dx = \left[-e^{-x^2/2}\right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$$

83. Let $u = 1 - e^x$, $du = -e^x\,dx$.

$$\int e^x\sqrt{1-e^x}\,dx = -\int (1-e^x)^{1/2}(-e^x)\,dx = -\frac{2}{3}(1-e^x)^{3/2} + C$$

84. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln(e^x + e^{-x}) + C$$

85. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x})\,dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln|e^x - e^{-x}| + C$$

86. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})\,dx$.

$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx = 2\int (e^x + e^{-x})^{-2}(e^x - e^{-x})\,dx$$
$$= \frac{-2}{e^x + e^{-x}} + C$$

87. $$\int \frac{5 - e^x}{e^{2x}}\,dx = \int 5e^{-2x}\,dx - \int e^{-x}\,dx$$
$$= -\frac{5}{2}e^{-2x} + e^{-x} + C$$

88. $$\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx = \int (e^x + 2 + e^{-x})\,dx$$
$$= e^x + 2x - e^{-x} + C$$

89. $$\int e^{\sin \pi x}\cos \pi x\,dx = \frac{1}{\pi}\int e^{\sin \pi x}(\pi \cos \pi x)\,dx$$
$$= \frac{1}{\pi}e^{\sin \pi x} + C$$

90. $$\int e^{\tan 2x}\sec^2 2x\,dx = \frac{1}{2}\int e^{\tan 2x}(2\sec^2 2x)\,dx$$
$$= \frac{1}{2}e^{\tan 2x} + C$$

91. $$\int e^{-x}\tan(e^{-x})\,dx = -\int [\tan(e^{-x})](-e^{-x})\,dx$$
$$= \ln|\cos(e^{-x})| + C$$

92. $$\int \ln(e^{2x-1})\,dx = \int (2x-1)\,dx$$
$$= x^2 - x + C$$

93. Let $u = ax^2$, $du = 2ax\,dx$. (Assume $a \neq 0$)

$$y = \int xe^{ax^2}\,dx$$
$$= \frac{1}{2a}\int e^{ax^2}(2ax)\,dx = \frac{1}{2a}e^{ax^2} + C$$

94. $$y = \int (e^x - e^{-x})^2\,dx$$
$$= \int (e^{2x} - 2 + e^{-2x})\,dx$$
$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$$

95. $$f'(x) = \int \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}(e^x - e^{-x}) + C_1$$
$$f'(0) = C_1 = 0$$
$$f(x) = \int \frac{1}{2}(e^x - e^{-x})\,dx = \frac{1}{2}(e^x + e^{-x}) + C_2$$
$$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$$
$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

96. $f'(x) = \int (\sin x + e^{2x})\,dx = -\cos x + \frac{1}{2}e^{2x} + C_1$

$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \Rightarrow C_1 = 1$

$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1$

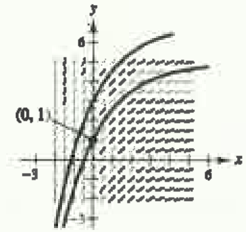
$f(x) = \int \left(-\cos x + \frac{1}{2}e^{2x} + 1\right) dx$

$= -\sin x + \frac{1}{4}e^{2x} + x + C_2$

$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \Rightarrow C_2 = 0$

$f(x) = x - \sin x + \frac{1}{4}e^{2x}$

97. (a)



(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

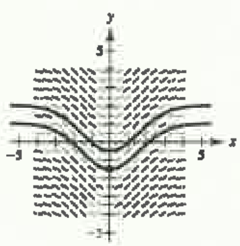
$y = \int 2e^{-x/2}\,dx = -4\int e^{-x/2}\left(-\frac{1}{2}dx\right)$

$= -4e^{-x/2} + C$

$(0, 1)$: $1 = -4e^0 + C = -4 + C \Rightarrow C = 5$

$y = -4e^{-x/2} + 5$

98. (a)



(b) $\frac{dy}{dx} = xe^{-0.2x^2}$, $\left(0, -\frac{3}{2}\right)$

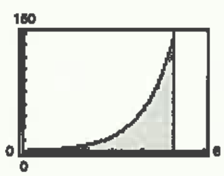
$y = \int xe^{-0.2x^2}\,dx = \frac{1}{-0.4}\int e^{-0.2x^2}(-0.4x)\,dx$

$= -\frac{1}{0.4}e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$

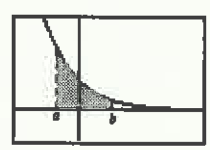
$\left(0, -\frac{3}{2}\right)$: $-\frac{3}{2} = -2.5e^0 + C = -2.5 + C \Rightarrow C = 1$

$y = -2.5e^{-0.2x^2} + 1$

99. $\int_0^5 e^x\,dx = \Big[e^x\Big]_0^5 = e^5 - 1 \approx 147.413$

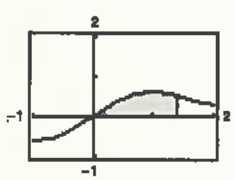


100. $\int_a^b e^{-x}\,dx = \Big[-e^{-x}\Big]_a^b = e^{-a} - e^{-b}$



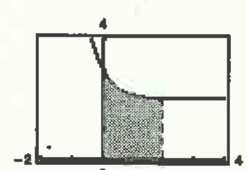
101. $\int_0^{\sqrt{2}} xe^{-(x^2/2)}dx = \Big[-e^{-(x^2/2)}\Big]_0^{\sqrt{2}}$

$= -e^{-1} + 1 \approx 0.632$



102. $\int_0^2 (e^{-2x} + 2)\,dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$



103. (a) $f(u-v) = e^{u-v} = (e^u)(e^{-v}) = \dfrac{e^u}{e^v} = \dfrac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$.

104. (a) $\displaystyle\int_0^4 \sqrt{x}e^x\,dx,\ n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing Utility: 92.7437

(b) $\displaystyle\int_0^2 2xe^{-x}\,dx,\ n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing Utility: 1.18799

105. $\displaystyle 0.0665\int_{48}^{60} e^{-0.0139(t-48)^2}\,dt$

Graphing Utility: 0.4772 = 47.72%

106. $\displaystyle\int_0^x e^t\,dt \ge \int_0^x 1\,dt$

$\Big[e^t\Big]_0^x \ge \Big[t\Big]_0^x$

$e^x - 1 \ge x \implies e^x \ge 1 + x$ for $x \ge 0$

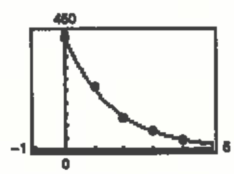
107.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$R = e^{-0.6155t+6.0609} = 428.78e^{-0.6155t}$

(b)



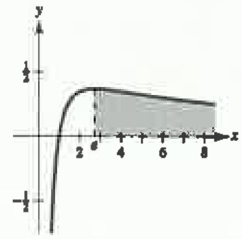
(c) $\displaystyle\int_0^4 R(t)\,dt = \int_0^4 428.78e^{-0.6155t}\,dt \approx 637.2$ liters

108. $\ln\dfrac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$\ln e^{a-b} = a - b$

Therefore, $\ln\dfrac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\dfrac{e^a}{e^b} = e^{a-b}$.

109. $f(x) = \dfrac{\ln x}{x}$



(a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e)$, $f'(x) > 0 \implies f$ is increasing.

On (e, ∞), $f'(x) < 0 \implies f$ is decreasing.

(b) For $e \le A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.

Section 5.5 Bases Other than e and Applications

1. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

2. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

3. $\log_7 1 = 0$

4. $\log_a \dfrac{1}{a} = \log_a 1 - \log_a a = -1$

5. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \dfrac{1}{3}$

$\log_3 \dfrac{1}{3} = -1$

6. (a) $27^{2/3} = 9$

$\log_{27} 9 = \dfrac{2}{3}$

(b) $16^{3/4} = 8$

$\log_{16} 8 = \dfrac{3}{4}$

7. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

8. (a) $\log_3 \frac{1}{9} = -2$

$3^{-2} = \frac{1}{9}$

(b) $49^{1/2} = 7$

$\log_{49} 7 = \frac{1}{2}$

9. (a) $\log_{10} 1000 = x$

$10^x = 1000$

$x = 3$

(b) $\log_{10} 0.1 = x$

$10^x = 0.1$

$x = -1$

10. (a) $\log_4 \frac{1}{64} = x$

$4^x = \frac{1}{64}$

$x = -3$

(b) $\log_5 25 = x$

$5^x = 25$

$x = 2$

11. (a) $\log_3 x = -1$

$3^{-1} = x$

$x = \frac{1}{3}$

(b) $\log_2 x = -4$

$2^{-4} = x$

$x = \frac{1}{16}$

12. (a) $\log_b 27 = 3$

$b^3 = 27$

$b = 3$

(b) $\log_b 125 = 3$

$b^3 = 125$

$b = 5$

13. (a)
$$x^2 - x = \log_5 25$$
$$x^2 - x = \log_5 5^2 = 2$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1 \text{ OR } x = 2$$

(b)
$$3x + 5 = \log_2 64$$
$$3x + 5 = \log_2 2^6 = 6$$
$$3x = 1$$
$$x = \tfrac{1}{3}$$

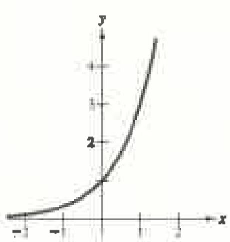
14. (a)
$$\log_3 x + \log_3(x - 2) = 1$$
$$\log_3[x(x - 2)] = 1$$
$$x(x - 2) = 3^1$$
$$x^2 - 2x - 3 = 0$$
$$(x + 1)(x - 3) = 0$$
$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b)
$$\log_{10}(x + 3) - \log_{10} x = 1$$
$$\log_{10}\frac{x + 3}{x} = 1$$
$$\frac{x + 3}{x} = 10^1$$
$$x + 3 = 10x$$
$$3 = 9x$$
$$x = \frac{1}{3}$$

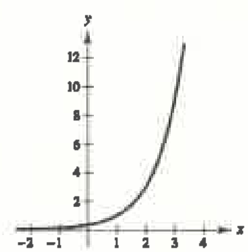
15. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



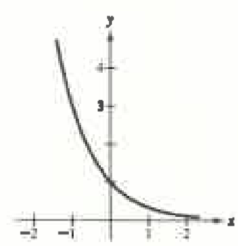
16. $y = 3^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



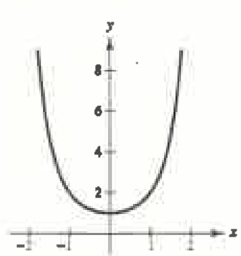
17. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



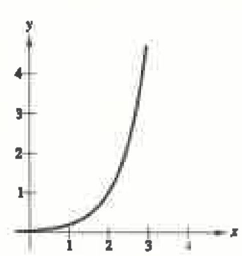
18. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



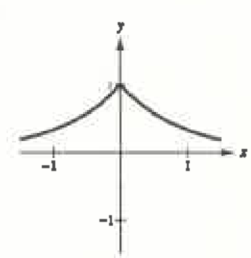
19. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



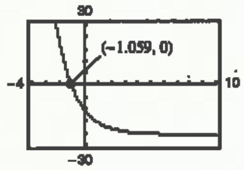
20. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



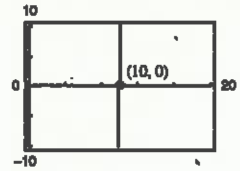
21. $g(x) = 6(2^{1-x}) - 25$

Zero: $x \approx -1.059$



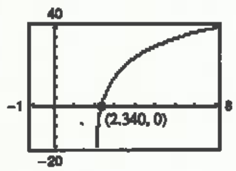
22. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



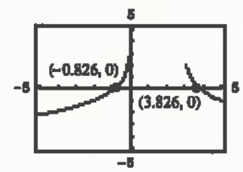
23. $h(s) = 32\log_{10}(s - 2) + 15$

Zero: $s \approx 2.340$



24. $g(x) = 1 - 2\log_{10}[x(x - 3)]$

Zeros: $x \approx -0.826, 3.826$

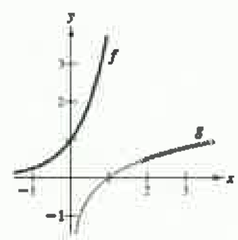


25. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1

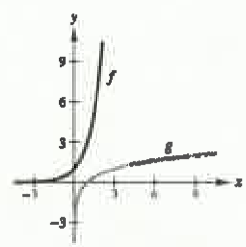


26. $f(x) = 3^x$

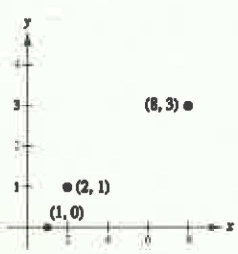
$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



27.



x	1	2	8
y	0	1	3

(a) y is an exponential function of x: False

(b) y is a logarithmic function of x: True; $y = \log_2 x$

(c) x is an exponential function of y: True, $2^y = x$

(d) y is a linear function of x: False

28. $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$10^y = x$

$f^{-1}(x) = 10^x$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 10{,}000 = \log_{10} 10^4 = 4$

If $1000 \le x \le 10{,}000$, then $3 \le f(x) \le 4$.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10$

$= \log_{10}(10x)$

x must have been increased by a factor of 10.

(f) $\log_{10}\left(\dfrac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$= 3n - n = 2n$

Thus, $x_1/x_2 = 10^{2n} = 100^n$.

29. $f(x) = 4^x$

$f'(x) = (\ln 4)\, 4^x$

30. $g(x) = 2^{-x}$

$g'(x) = -(\ln 2)\, 2^{-x}$

31. $y = 5^{x-2}$

$\dfrac{dy}{dx} = (\ln 5)\, 5^{x-2}$

32. $y = x(7^{-3x})$

$\dfrac{dy}{dx} = x[-3(\ln 7)\, 7^{-3x}] + 7^{-3x}$

$= 7^{-3x}[-3x(\ln 7) + 1]$

$= 7^{-3x}(1 - 3x \ln 7)$

33. $g(t) = t^2\, 2^t$

$g'(t) = t^2 (\ln 2)\, 2^t + (2t)\, 2^t$

$= t2^t (t \ln 2 + 2)$

$= 2^t t(2 + t \ln 2)$

34. $f(t) = \dfrac{3^{2t}}{t}$

$f'(t) = \dfrac{t(2 \ln 3)\, 3^{2t} - 3^{2t}}{t^2}$

$= \dfrac{3^{2t}(2t \ln 3 - 1)}{t^2}$

35. $h(\theta) = 2^{-\theta} \cos \pi\theta$

$h'(\theta) = 2^{-\theta}(-\pi \sin \pi\theta) - (\ln 2) 2^{-\theta} \cos \pi\theta$

$= -2^{-\theta}[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta]$

36. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$g'(\alpha) = 5^{-\alpha/2}\, 2 \cos 2\alpha - \frac{1}{2}(\ln 5)\, 5^{-\alpha/2} \sin 2\alpha$

37. $y = \log_3 x$

$\dfrac{dy}{dx} = \dfrac{1}{x \ln 3}$

38. $y = \log_{10}(2x) = \log_{10} 2 + \log_{10} x$

$\dfrac{dy}{dx} = 0 + \dfrac{1}{x \ln 10} = \dfrac{1}{x \ln 10}$

39. $f(x) = \log_2 \dfrac{x^2}{x-1}$

$= 2 \log_2 x - \log_2 (x - 1)$

$f'(x) = \dfrac{2}{x \ln 2} - \dfrac{1}{(x-1) \ln 2}$

$= \dfrac{x - 2}{(\ln 2)x(x - 1)}$

40. $h(x) = \log_3 \dfrac{x\sqrt{x-1}}{2}$

$= \log_3 x + \dfrac{1}{2} \log_3 (x - 1) - \log_3 2$

$h'(x) = \dfrac{1}{x \ln 3} + \dfrac{1}{2} \cdot \dfrac{1}{(x-1) \ln 3} - 0$

$= \dfrac{1}{\ln 3}\left[\dfrac{1}{x} + \dfrac{1}{2(x-1)}\right]$

$= \dfrac{1}{\ln 3}\left[\dfrac{3x - 2}{2x(x-1)}\right]$

41. $y = \log_5 \sqrt{x^2 - 1} = \dfrac{1}{2} \log_5 (x^2 - 1)$

$\dfrac{dy}{dx} = \dfrac{1}{2} \cdot \dfrac{2x}{(x^2 - 1)\ln 5} = \dfrac{x}{(x^2 - 1)\ln 5}$

42. $$y = \log_{10}\frac{x^2 - 1}{x}$$
$$= \log_{10}(x^2 - 1) - \log_{10} x$$
$$\frac{dy}{dx} = \frac{2x}{(x^2 - 1)\ln 10} - \frac{1}{x\ln 10}$$
$$= \frac{1}{\ln 10}\left[\frac{2x}{x^2 - 1} - \frac{1}{x}\right]$$
$$= \frac{1}{\ln 10}\left[\frac{x^2 + 1}{x(x^2 - 1)}\right]$$

43. $$g(t) = \frac{10\log_4 t}{t} = \frac{10}{\ln 4}\left(\frac{\ln t}{t}\right)$$
$$g'(t) = \frac{10}{\ln 4}\left[\frac{t(1/t) - \ln t}{t^2}\right]$$
$$= \frac{10}{t^2\ln 4}[1 - \ln t] = \frac{5}{t^2\ln 2}(1 - \ln t)$$

44. $$f(t) = t^{3/2}\log_2\sqrt{t + 1} = t^{3/2}\frac{1}{2}\frac{\ln(t + 1)}{\ln 2}$$
$$f'(t) = \frac{1}{2\ln 2}\left[t^{3/2}\frac{1}{t + 1} + \frac{3}{2}t^{1/2}\ln(t + 1)\right]$$

45. $$y = x^{2/x}$$
$$\ln y = \frac{2}{x}\ln x$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) = \frac{2}{x^2}(1 - \ln x)$$
$$\frac{dy}{dx} = \frac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$$

46. $$y = x^{x-1}$$
$$\ln y = (x - 1)(\ln x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x - 1)\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = y\left[\frac{x - 1}{x} + \ln x\right]$$
$$= x^{x-2}(x - 1 + x\ln x)$$

47. $$y = (x - 2)^{x+1}$$
$$\ln y = (x + 1)\ln(x - 2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x + 1)\left(\frac{1}{x - 2}\right) + \ln(x - 2)$$
$$\frac{dy}{dx} = y\left[\frac{x + 1}{x - 2} + \ln(x - 2)\right]$$
$$= (x - 2)^{x+1}\left[\frac{x + 1}{x - 2} + \ln(x - 2)\right]$$

48. $$y = (1 + x)^{1/x}$$
$$\ln y = \frac{1}{x}\ln(1 + x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{1}{1 + x}\right) + \ln(1 + x)\left(-\frac{1}{x^2}\right)$$
$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$
$$= \frac{(1 + x)^{1/x}}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$

49. $f(x) = \log_2 x \implies f'(x) = \dfrac{1}{x\ln 2}$

$g(x) = x^x \implies g'(x) = x^x(1 + \ln x)$

[**Note:** Let $y = g(x)$. Then: $\ln y = \ln x^x = x\ln x$
$$\frac{1}{y}y' = x\cdot\frac{1}{x} + \ln x$$
$$y' = y(1 + \ln x)$$
$$y' = x^x(1 + \ln x) = g'(x).]$$

$h(x) = x^2 \implies h'(x) = 2x$

$k(x) = 2^x \implies k'(x) = (\ln 2)2^x$

From greatest to smallest rate of growth:
$g(x), k(x), h(x), f(x)$

50. $f(x) = a^x$

(a) $f(u + v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

51. $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$

$\approx \$40.64$

(b) $\dfrac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$: $\dfrac{dC}{dt} \approx 0.051P$

When $t = 8$: $\dfrac{dC}{dt} \approx 0.072P$

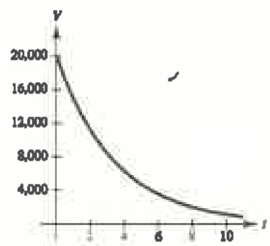
(c) $\dfrac{dC}{dt} = (\ln 1.05)[P(1.05)^t]$

$= (\ln 1.05)C(t)$

The constant of proportionality is ln 1.05.

52. $V(t) = 20{,}000\left(\dfrac{3}{4}\right)^t$

(a)



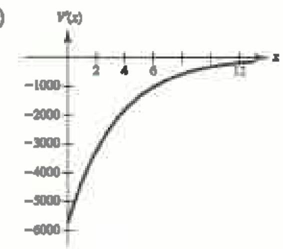
$V(2) = 20{,}000\left(\dfrac{3}{4}\right)^2 = \$11{,}250$

(b) $\dfrac{dV}{dt} = 20{,}000\left(\ln\dfrac{3}{4}\right)\left(\dfrac{3}{4}\right)^t$

When $t = 1$: $\dfrac{dV}{dt} \approx -4315.23$

When $t = 4$: $\dfrac{dV}{dt} \approx -1820.49$

(c)



Horizontal asymptote: $v' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach $0.

53. $P = \$1000,\ r = 3\tfrac{1}{2}\% = 0.035,\ t = 10$

$A = 1000\left(1 + \dfrac{0.035}{n}\right)^{10n}$

$A = 1000e^{(0.035)(10)} = 1419.07$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

54. $P = \$2500,\ r = 6\% = 0.06,\ t = 20$

$A = 2500\left(1 + \dfrac{0.06}{n}\right)^{20n}$

$A = 2500e^{(0.06)(20)} = 8300.29$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

55. $P = \$1000,\ r = 5\% = 0.05,\ t = 30$

$A = 1000\left(1 + \dfrac{0.05}{n}\right)^{30n}$

$A = 1000e^{(0.05)30} = 4481.69$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

56. $P = \$2500,\ r = 5\% = 0.05,\ t = 40$

$A = 2500\left(1 + \dfrac{0.05}{n}\right)^{40n}$

$A = 2500e^{(0.05)(40)} = 18{,}472.64$

n	1	2	4	12	365	Continuous
A	17,599.97	18,023.92	18,245.05	18,396.04	18,470.11	18,472.64

57. $100{,}000 = Pe^{0.05t} \implies P = 100{,}000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,583.53	8208.50

58. $100{,}000 = Pe^{0.06t} \implies P = 100{,}000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

59. $100{,}000 = P\left(1 + \dfrac{0.05}{12}\right)^{12t} \implies P = 100{,}000\left(1 + \dfrac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

60. $100{,}000 = P\left(1 + \dfrac{0.07}{365}\right)^{365t} \implies P = 100{,}000\left(1 + \dfrac{0.07}{365}\right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

61. (a) $A = 20{,}000\left(1 + \dfrac{0.06}{365}\right)^{(365)(8)} \approx \$32{,}320.21$

(b) $A = \$30{,}000$

(c) $A = 8000\left(1 + \dfrac{0.06}{365}\right)^{(365)(8)} + 20{,}000\left(1 + \dfrac{0.06}{365}\right)^{(365)(4)}$

$\approx \$12{,}928.09 + 25{,}424.48 = \$38{,}352.57$

(d) $A = 9000\left[\left(1 + \dfrac{0.06}{365}\right)^{(365)(8)} + \left(1 + \dfrac{0.06}{365}\right)^{(365)(4)} + 1\right]$

$\approx \$34{,}985.11$

Take option (c).

62. Let $P = \$100$, $0 \le t \le 20$.

(a) $A = 100e^{0.03t}$

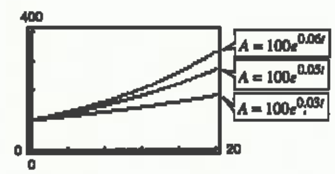
$A(20) \approx 182.21$

(b) $A = 100e^{0.05t}$

$A(20) \approx 271.83$

(c) $A = 100e^{0.06t}$

$A(20) \approx 332.01$



63. (a) $\lim_{t\to\infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

(b) $V' = \dfrac{322.27}{t^2}e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

64. (a) $\lim_{n\to\infty} \dfrac{0.83}{1 + e^{-0.2n}} = 0.83 = 83\%$

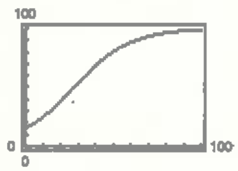
(b) $P' = \dfrac{0.166e^{-0.2n}}{(1 + e^{-0.2n})^2}$

$P'(3) \approx 0.038$

$P'(10) \approx 0.017$

65. $y = \dfrac{300}{3 + 17e^{-0.0625x}}$

(a)



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

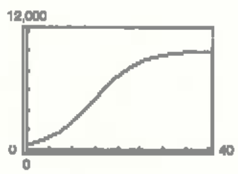
(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$$

$$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$$

$17e^{-0.0625x} - 3 = 0 \implies x \approx 27.8$ or 27,800 egg masses.

66. (a)



(b) Limiting size: 10,000 fish

(c) $p(t) = \dfrac{10,000}{1 + 19e^{-t/5}}$

$$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{t/5})^2}\left(\frac{19}{5}\right)(10,000)$$

$$= \frac{38,000e^{-t/5}}{(1 + 19e^{-t/5})^2}$$

$p'(1) \approx 113.5$ fish/month

$p'(10) \approx 403.2$ fish/month

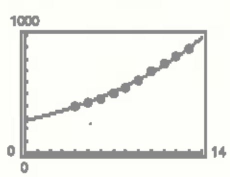
(d) $p''(t) = -\dfrac{38,000}{5}(e^{-t/5})\left[\dfrac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3}\right] = 0$

$19e^{-t/5} = 1$

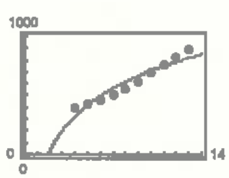
$\dfrac{t}{5} = \ln 19$

$t = 5 \ln 19 \approx 14.72$

67. (a) $y = (271.92)(1.096)^x = (271.92)e^{0.0918x}$



(b) $y = -254.08 + 418.41 \ln x$



(c) The exponential model is better.

(d) Exponential model:

$$\frac{dy}{dx} = (2171.92)(0.0918)(e^{0.0918(20)})$$

$$\approx 157$$

Logarithmic model: $\dfrac{dy}{dx} = (418.41)\dfrac{1}{20} \approx 20.9$

Logarithmic model would be better.

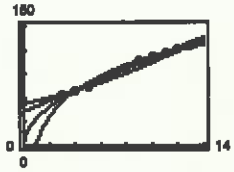
68. (a) $y_1 = 6.65x + 43.31$

$y_2 = 4.75 + 46.19 \ln x$

$y_3 = (52.97)(1.07)^x$

$y_4 = (34.31)(x^{0.51})$

(b)



The y_4 model seems best.

(c) The slope 6.65 is the annual rate of change in the amount given to philanthropy.

(d) $y_1{}' = 6.65$

$y_2{}' = \dfrac{46.19}{12} = 3.85$

$y_3{}' = (52.97)\ln(1.07)(1.07)^{12} = 7.54$

$y_4{}' = (34.31)(0.51)(12)^{0.51-1} = 5.18$

The third model is increasing most rapidly.

69. $\displaystyle\int 3^x\,dx = \frac{3^x}{\ln 3} + C$

70. $\displaystyle\int 4^{-x}\,dx = -\frac{4^{-x}}{\ln 4} + C$

71. $\displaystyle\int_{-1}^{2} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2}$

$\displaystyle= \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right]$

$\displaystyle= \frac{7}{2\ln 2} = \frac{7}{\ln 4}$

72. $\displaystyle\int_{-2}^{0} (3^3 - 5^2)\,dx = \int_{-2}^{0} (27 - 25)\,dx$

$\displaystyle= \int_{-2}^{0} 2\,dx$

$\displaystyle= \Big[2x\Big]_{-2}^{0} = 4$

73. $\displaystyle\int x5^{-x^2}\,dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\,dx$

$\displaystyle= -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$

$\displaystyle= \frac{-1}{2\ln 5}(5^{-x^2}) + C$

74. $\displaystyle\int (3 - x)\,7^{(3-x)^2}\,dx = -\frac{1}{2}\int -2(3 - x)\,7^{(3-x)^2}\,dx$

$\displaystyle= -\frac{1}{2\ln 7}[7^{(3-x)^2}] + C$

75. $\displaystyle\int \frac{3^{2x}}{1 + 3^{2x}}\,dx,\ u = 1 + 3^{2x},\ du = 2(\ln 3)3^{2x}\,dx$

$\displaystyle\frac{1}{2\ln 3}\int \frac{(2\ln 3)3^{2x}}{1 + 3^{2x}}\,dx = \frac{1}{2\ln 3}\ln(1 + 3^{2x}) + C$

76. $\displaystyle\int 2^{\sin x}\cos x\,dx,\ u = \sin x,\ du = \cos x\,dx$

$\displaystyle\frac{1}{\ln 2}2^{\sin x} + C$

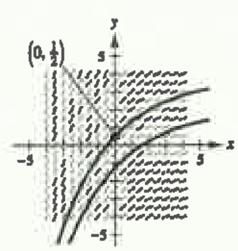
77. $\displaystyle\frac{dy}{dx} = 0.4^{x/3},\ \left(0, \frac{1}{2}\right)$

$\displaystyle y = \int 0.4^{x/3}\,dx = 3\int 0.4^{x/3}\left(\frac{1}{3}\,dx\right)$

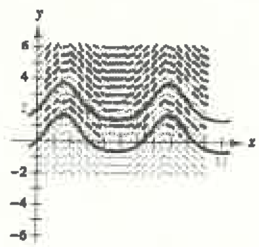
$\displaystyle= \frac{3}{\ln 0.4}0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C$

$\displaystyle\left(0, \frac{1}{2}\right):\ \frac{1}{2} = 3(\ln 2.5) + C \Rightarrow C = \frac{1}{2} - 3\ln 2.5$

$\displaystyle y = 3\ln 2.5(0.4)^{x/3} + \frac{1}{2} - 3\ln 2.5 = \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}$



78. (a)



(b) $\dfrac{dy}{dx} = e^{\sin x}\cos x \qquad (\pi, 2)$

$$y = \int e^{\sin x}\cos x\,dx = e^{\sin x} + C$$

$(\pi, 2)$: $2 = e^{\sin \pi} + C = 1 + C \Rightarrow C = 1$

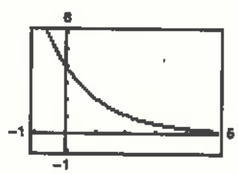
$$y = e^{\sin x} + 1$$

79. (a) $\displaystyle\int_0^4 f(t)\,dt \approx 5.67$

$\displaystyle\int_0^4 g(t)\,dt \approx 5.67$

$\displaystyle\int_0^4 h(t)\,dt \approx 5.67$

(b)



(c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

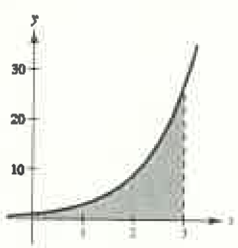
$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4\left[e^{-0.653886}\right]^t = 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t = 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

80. $A = \displaystyle\int_0^3 3^x\,dx = \left[\frac{3^x}{\ln 3}\right]_0^3 = \frac{26}{\ln 3} \approx 23.666$



81. $P = \displaystyle\int_0^{10} 2000e^{-0.06t}\,dt$

$$= \left[\frac{2000}{-0.06}e^{-0.06t}\right]_0^{10}$$

$\approx \$15{,}039.61$

82.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$	2	2.594	2.705	2.718	2.718

83.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0$, $y = 1200 \Rightarrow C = 1200$.

$y = 1200(k^t)$

$\dfrac{720}{1200} = 0.6,\ \dfrac{432}{720} = 0.6,\ \dfrac{259.20}{432} = 0.6,\ \dfrac{155.52}{259.20} = 0.6$

Let $k = 0.6$.

$y = 1200(0.6)^t$

84.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$y = C(k^t)$

When $t = 0$, $y = 600 \Rightarrow C = 600$.

$y = 600(k^t)$

$\dfrac{630}{600} = 1.05,\ \dfrac{661.50}{630} = 1.05,\ \dfrac{694.58}{661.50} \approx 1.05,\ \dfrac{729.30}{694.58} \approx 1.0$

Let $k = 1.05$.

$y = 600(1.05)^t$

85. False. e is an irrational number.

86. True.

$$f(e^{n+1}) - f(e^n) = \ln e^{n+1} - \ln e^n$$
$$= n + 1 - n$$
$$= 1$$

87. True.

$$f(g(x)) = 2 + e^{\ln(x-2)}$$
$$= 2 + x - 2 = x$$
$$g(f(x)) = \ln(2 + e^x - 2)$$
$$= \ln e^x = x$$

88. True.

$$\frac{d^n y}{dx^n} = Ce^x$$
$$= y \text{ for } n = 1, 2, 3, \ldots$$

89. True.

$$\frac{d}{dx}[e^x] = e^x \text{ and } \frac{d}{dx}[e^{-x}] = -e^{-x}$$

$e^x = e^{-x}$ when $x = 0$.

$$(e^0)(-e^{-0}) = -1$$

90. True.

$f(x) = g(x)e^x = 0 \Rightarrow$

$g(x) = 0$ since $e^x > 0$ for all x.

91.

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}dt \Rightarrow \frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int \frac{8}{25}dt \Rightarrow$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1 e^{(2/5)t}$$

$$y(0) = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \frac{y}{(5/4) - y}$$

$$\Rightarrow 4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \Rightarrow 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$\Rightarrow y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

92. $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \sin x\left(\frac{1}{x}\right) + \cos x \cdot \ln x$$

$$y' = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \cdot \ln x\right]$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$y' = \left(\frac{\pi}{2}\right)^{\sin(\pi/2)}\left[\frac{\sin(\pi/2)}{\pi/2} + \cos\left(\frac{\pi}{2}\right)\ln\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{2}{\pi} + 0\right] = 1$$

Tangent line: $y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$$y = x$$

Section 5.6 Differential Equations: Growth and Decay

1. $$\begin{aligned} y' &= \frac{5x}{y} \\ yy' &= 5x \\ \int yy'\,dx &= \int 5x\,dx \\ \int y\,dy &= \int 5x\,dx \\ \frac{1}{2}y^2 &= \frac{5}{2}x^2 + C_1 \\ y^2 - 5x^2 &= C \end{aligned}$$

2. $$\begin{aligned} y' &= \frac{\sqrt{x}}{2y} \\ 2yy' &= \sqrt{x} \\ \int 2yy'\,dx &= \int \sqrt{x}\,dx \\ \int 2y\,dy &= \int \sqrt{x}\,dx \\ y^2 &= \frac{2}{3}x^{3/2} + C_1 \\ 3y^2 - 2x^{3/2} &= C \end{aligned}$$

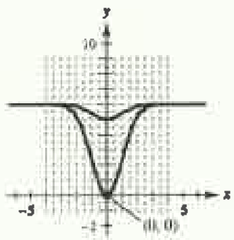
3. $$\begin{aligned} y' &= \sqrt{x}\,y \\ \frac{y'}{y} &= \sqrt{x} \\ \int \frac{y'}{y}\,dx &= \int \sqrt{x}\,dx \\ \int \frac{dy}{y} &= \int \sqrt{x}\,dx \\ \ln y &= \frac{2}{3}x^{3/2} + C_1 \\ y &= e^{(2/3)x^{3/2}+C_1} \\ &= e^{C_1}e^{(2/3)x^{3/2}} \\ &= Ce^{(2/3)x^{3/2}} \end{aligned}$$

4. $$\begin{aligned} y' &= x(1+y) \\ \frac{y'}{1+y} &= x \\ \int \frac{y'}{1+y}\,dx &= \int x\,dx \\ \int \frac{dy}{1+y} &= \int x\,dx \\ \ln(1+y) &= \frac{x^2}{2} + C_1 \\ 1 + y &= e^{(x^2/2)+C_1} \\ y &= e^{C_1}e^{x^2/2} - 1 \\ &= Ce^{x^2/2} - 1 \end{aligned}$$

5. $$\begin{aligned} (1+x^2)y' - 2xy &= 0 \\ y' &= \frac{2xy}{1+x^2} \\ \frac{y'}{y} &= \frac{2x}{1+x^2} \\ \int \frac{y'}{y}\,dx &= \int \frac{2x}{1+x^2}\,dx \\ \int \frac{dy}{y} &= \int \frac{2x}{1+x^2}\,dx \\ \ln y &= \ln(1+x^2) + C_1 \\ \ln y &= \ln(1+x^2) + \ln C \\ \ln y &= \ln C(1+x^2) \\ y &= C(1+x^2) \end{aligned}$$

6. $$\begin{aligned} xy + y' &= 100x \\ y' &= 100x + xy = x(100 - y) \\ \frac{y'}{100-y} &= x \\ \int \frac{y'}{100-y}\,dx &= \int x\,dx \\ \int \frac{1}{100-y}\,dy &= \int x\,dx \\ -\ln(100-y) &= \frac{x^2}{2} + C_1 \\ \ln(100-y) &= -\frac{x^2}{2} - C_1 \\ 100 - y &= e^{-(x^2/2)-C_1} \\ -y &= e^{-C_1}e^{-x^2/2} - 100 \\ y &= 100 - Ce^{-x^2/2} \end{aligned}$$

7. (a)



(b) $\frac{dy}{dx} = x(6 - y), \quad (0, 0)$

$$\frac{dy}{y - 6} = -x$$

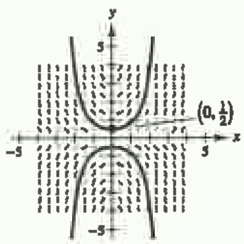
$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0)\text{: } 0 = 6 + C_1 \Rightarrow C_1 = -6 \Rightarrow y = 6 - 6e^{-x^2/2}$$

8. (a)



(b) $\frac{dy}{dx} = xy, \left(0, \frac{1}{2}\right)$

$$\frac{dy}{y} = x\,dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right)\text{: } \frac{1}{2} = C_1 e^0 \Rightarrow C_1 = \frac{1}{2} \Rightarrow y = \frac{1}{2} e^{x^2/2}$$

9. $\frac{dQ}{dt} = \frac{k}{t^2}$

$$\int \frac{dQ}{dt}\,dt = \int \frac{k}{t^2}\,dt$$

$$\int dQ = -\frac{k}{t} + C$$

$$Q = -\frac{k}{t} + C$$

10. $\frac{dP}{dt} = k(10 - t)$

$$\int \frac{dP}{dt}\,dt = \int k(10 - t)\,dt$$

$$\int dP = -\frac{k}{2}(10 - t)^2 + C$$

$$P = -\frac{k}{2}(10 - t)^2 + C$$

11. $\frac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds}\,ds = \int k(250 - s)\,ds$$

$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$

$$N = -\frac{k}{2}(250 - s)^2 + C$$

12. $\frac{dy}{dx} = kx(L - y)$

$$\frac{1}{L - y}\frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y}\frac{dy}{dx}\,dx = \int kx\,dx$$

$$\int \frac{1}{L - y}\,dy = \frac{kx^2}{2} + C_1$$

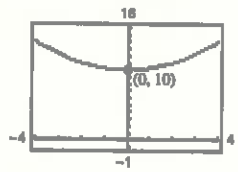
$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1} e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

13. $\frac{dy}{dt} = \frac{1}{2}t, \; (0, 10)$



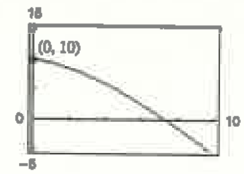
$$\int dy = \int \frac{1}{2}t\, dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \implies C = 10$$

$$y = \frac{1}{4}t^2 + 10$$

14. $\frac{dy}{dt} = -\frac{3}{4}\sqrt{t}, \; (0, 10)$



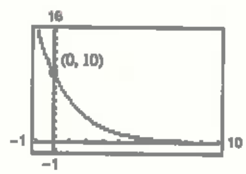
$$\int dy = \int -\frac{3}{4}\sqrt{t}\, dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \implies C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$

15. $\frac{dy}{dt} = -\frac{1}{2}y, \; (0, 10)$



$$\int \frac{dy}{y} = \int -\frac{1}{2}\, dt$$

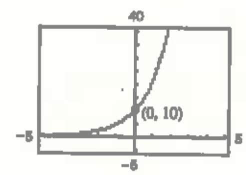
$$\ln y = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2)+C_1} = e^{C_1}e^{-t/2} = Ce^{-t/2}$$

$$10 = Ce^0 \implies C = 10$$

$$y = 10e^{-t/2}$$

16. $\frac{dy}{dt} = \frac{3}{4}y, \; (0, 10)$



$$\int \frac{dy}{y} = \int \frac{3}{4}\, dt$$

$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1}e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \implies C = 10$$

$$y = 10e^{3t/4}$$

17. $y = Ce^{kt}, \; \left(0, \frac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5} \approx 0.4605$$

$$y = \frac{1}{2}e^{0.4605t}$$

18. $y = Ce^{kt}, \; (0, 4), \left(5, \frac{1}{2}\right)$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

19. $y = Ce^{kt}$, $(1, 1)$, $(5, 5)$

$$1 = Ce^{k}$$
$$5 = Ce^{5k}$$
$$5Ce^{k} = Ce^{5k}$$
$$5e^{k} = e^{5k}$$
$$5 = e^{4k}$$
$$k = \frac{\ln 5}{4} \approx 0.4024$$
$$y = Ce^{0.4024t}$$
$$1 = Ce^{0.4024}$$
$$C \approx 0.6687$$
$$y = 0.6687e^{0.4024t}$$

20. $y = Ce^{kt}$, $\left(3, \frac{1}{2}\right)$, $(4, 5)$

$$\frac{1}{2} = Ce^{3k}$$
$$5 = Ce^{4k}$$
$$2Ce^{3k} = \frac{1}{5}Ce^{4k}$$
$$10e^{3k} = e^{4k}$$
$$10 = e^{k}$$
$$k = \ln 10 \approx 2.3026$$
$$y = Ce^{2.3026t}$$
$$5 = Ce^{2.3026(4)}$$
$$C \approx 0.0005$$
$$y = 0.0005e^{2.3026t}$$

21. Since the intial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10{,}000$, $y = 10e^{[\ln(1/2)/1620](10{,}000)} \approx 0.14$ gram.

22. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $1.5 = Ce^{[\ln(1/2)/1620](1000)} \implies C \approx 2.30$ which implies that the intitial quantity is 2.30 grams. When $t = 10{,}000$, we have $y = 2.30e^{[\ln(1/2)/1620](10{,}000)} \approx 0.03$ gram.

23. Since $y = Ce^{[\ln(1/2)/5730]t}$, we have $2.0 = Ce^{[\ln(1/2)/5730](10{,}000)} \implies C \approx 6.70$ which implies that the initial quantity is 6.70 grams. When $t = 1000$, we have $y = 6.70e^{[\ln(1/2)/5730](1000)} \approx 5.94$ grams.

24. Since the initial quantity is 3.0 grams, we have $y = 3.0e^{[\ln(1/2)/5730]t}$. When $t = 1000$, $y = 3.0e^{[\ln(1/2)/5730](1000)} \approx 2.66$ grams. When $t = 10{,}000$, $y = 3.0e^{[\ln(1/2)/5730](10{,}000)} \approx 0.89$ gram.

25. Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24{,}360](1000)} \implies C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10{,}000$, $y = 2.16e^{[\ln(1/2)/24{,}360](10{,}000)} \approx 1.63$ grams.

26. Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $0.4 = Ce^{[\ln(1/2)/24{,}360](10{,}000)} \implies C \approx 0.53$ which implies that the initial quantity is 0.53 gram. When $t = 1000$, we have $y = 0.53e^{[\ln(1/2)/24{,}360](1000)} \approx 0.52$ gram.

27. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{1620k}$$
$$k = \frac{-\ln 2}{1620}$$
$$y = y_0e^{-(\ln 2)t/1620}$$

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$.
Therefore, 95.81% of the present amount still exists.

28. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{5730k}$$
$$k = -\frac{\ln 2}{5730}$$
$$0.15y_0 = y_0e^{(-\ln 2/5730)t}$$
$$\ln 0.15 = -\frac{(\ln 2)t}{5730}$$
$$t = -\frac{5730 \ln 0.15}{\ln 2} \approx 15{,}682.8 \text{ years.}$$

29. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$2 = e^{0.06t}$$
$$\ln 2 = 0.06t$$
$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \$1822.12$

30. Since $A = 20{,}000e^{0.055t}$, the time to double is given by $40{,}000 = 20{,}000e^{0.055t}$ and we have

$$2 = e^{0.055t}$$
$$\ln 2 = 0.055t$$
$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years: $A = 20{,}000e^{(0.055)(10)} \approx \$34{,}665.06$

31. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$
$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

32. Since $A = 10{,}000e^{rt}$ and $A = 20{,}000$ when $t = 5$, we have the following.

$$20{,}000 = 10{,}000e^{5r}$$
$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

Amount after 10 years: $A = 10{,}000e^{[(\ln 2)/5](10)} = \$40{,}000$

33. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$
$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$
$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

34. Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$
$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$
$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

35. $500{,}000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$
$$\approx \$112{,}087.09$$

36. $500{,}000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 500{,}000(1.005)^{-480} \approx \$45{,}631.04$$

37. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$
$$\ln 2 = t \ln 1.07$$
$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$
$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$
$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$
$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$
$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$
$$\ln 2 = 0.07t$$
$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

38. (a) $2000 = 1000(1 + 0.065)^t$

$2 = 1.065^t$

$\ln 2 = t \ln 1.065$

$t = \dfrac{\ln 2}{\ln 1.065} \approx 11.01$ years

(b) $2000 = 1000\left(1 + \dfrac{0.065}{12}\right)^{12t}$

$2 = \left(1 + \dfrac{0.065}{12}\right)^{12t}$

$\ln 2 = 12t \ln\left(1 + \dfrac{0.065}{12}\right)$

$t = \dfrac{\ln 2}{12 \ln(1 + (0.065/12))} \approx 10.69$ years

(c) $2000 = 1000\left(1 + \dfrac{0.065}{365}\right)^{365t}$

$2 = \left(1 + \dfrac{0.065}{365}\right)^{365t}$

$\ln 2 = 365t \ln\left(1 + \dfrac{0.065}{365}\right)$

$t = \dfrac{\ln 2}{365 \ln(1 + (0.065/365))} \approx 10.66$ years

(d) $2000 = 1000e^{0.065t}$

$2 = e^{0.065t}$

$\ln 2 = 0.065t$

$t = \dfrac{\ln 2}{0.065} \approx 10.66$ years

39. Let $t = 0$ represent 1990.

$y = Ce^{kt}$, $(0, 4.22)$, $(10, 6.49)$

$C = 4.22$

$6.49 = 4.22e^{10k}$

$k = \dfrac{\ln(6.49/4.22)}{10} \approx 0.0430$

$y \approx 4.22e^{0.0430t}$

When $t = 20$ (for 2010), $y \approx 9.97$ million.

40. Let $t = 0$ represent 1990.

$y = Ce^{kt}$, $(0, 2.30)$, $(10, 2.65)$

$C = 2.30$

$2.65 = 2.30e^{10k}$

$k = \dfrac{\ln(2.65/2.30)}{10} \approx 0.0142$

$y \approx 2.30e^{0.0142t}$

When $t = 20$ (for 2010), $y \approx 3.06$ million.

41. Let $t = 0$ represent 1990.

$y = Ce^{kt}$, $(0, 3.0)$, $(10, 2.74)$

$C = 3.0$

$2.74 = 3.0e^{10k}$

$k = \dfrac{\ln(2.74/3.0)}{10} \approx -0.0091$

$y \approx 3.0e^{-0.0091t}$

When $t = 20$ (for 2010), $y \approx 2.51$ million.

42. Let $t = 0$ represent 1990.

$y = Ce^{kt}$, $(0, 9.17)$, $(10, 8.57)$

$C = 9.17$

$8.57 = 9.17e^{10k}$

$k = \dfrac{\ln(8.57/9.17)}{10} \approx -0.0068$

$y \approx 9.17e^{-0.0068t}$

When $t = 20$ (for 2010), $y \approx 8.0$ million.

43. (a) k; the larger the value of k, the greater the rate of growth of the population.

(b) k; if $k > 0$, there is growth. If $k < 0$, decrease.

44. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$400 = 100.1596(1.2455)^t$

$1.2455^t = \dfrac{400}{100.1596} = 3.9936$

$t \ln 1.2455 = \ln 3.9936$

$t = \dfrac{\ln 3.9936}{\ln 1.2455} \approx 6.3$ hours.

45. $P = Ce^{kx}$, $(0, 760)$, $(1000, 672.71)$

$C = 760$

$672.71 = 760e^{1000x}$

$x = \dfrac{\ln(672.71/760)}{1000} \approx -0.000122$

$P \approx 760e^{-0.000122x}$

When $x = 3000$, $P \approx 527.06$ mm Hg.

46. $y = Ce^{kt}$, $(0, 742{,}000)$, $(2, 632{,}000)$

$C = 742{,}000$

$632{,}000 = 742{,}000e^{2k}$

$k = \dfrac{\ln(632/742)}{2} \approx -0.0802$

$y \approx 742{,}000e^{-0.0802t}$

When $t = 3$, $y \approx \$583{,}327.77$.

47. (a) $19 = 30(1 - e^{20k})$

$30e^{20k} = 11$

$k = \dfrac{\ln(11/30)}{20} \approx -0.0502$

$N \approx 30(1 - e^{-0.0502t})$

(b) $25 = 30(1 - e^{-0.0502t})$

$e^{-0.0502t} = \dfrac{1}{6}$

$t = \dfrac{-\ln 6}{-0.0502} \approx 36$ days

48. (a) $20 = 30(1 - e^{30k})$

$30e^{30k} = 10$

$k = \dfrac{\ln(1/3)}{30} = \dfrac{-\ln 3}{30} \approx -0.0366$

$N \approx 30(1 - e^{-0.0366t})$

(b) $25 = 30(1 - e^{-0.0366t})$

$e^{-0.0366t} = \dfrac{1}{6}$

$t = \dfrac{-\ln 6}{-0.0366} \approx 49$ days

49. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$5 = Ce^{k}$

$\lim_{t \to \infty} Ce^{k/t} = C = 30$

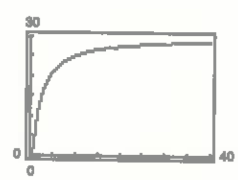
$5 = 30e^{k}$

$k = \ln \frac{1}{6} \approx -1.7918$

$S \approx 30e^{-1.7918/t}$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)



50. $S = 30(1 - e^{kt})$

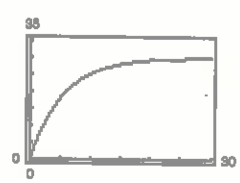
(a) $5 = 30(1 - e^{k(1)}) \implies k = \ln \frac{5}{6} \approx -0.1823$

$S \approx 30(1 - e^{-0.1823t})$

(b) 30,000 units $\left(\lim_{x \to \infty} S = 30\right)$

(c) When $t = 5$, $S \approx 17.9424$ which is 17,942 units.

(d)



51. $A(t) = V(t)e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}}e^{-0.10t} = 100{,}000e^{0.8\sqrt{t} - 0.10t}$

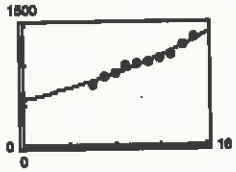
$\dfrac{dA}{dt} = 100{,}000\left(\dfrac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t} - 0.10t} = 0$ when 16.

The timber should be harvested in the year 2014, (1998 + 16). **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \le x \le 30$ and $0 \le y \le 600{,}000$.

52. (a) $R = 572.85(1.05766)^t = 572.85e^{0.056t}$

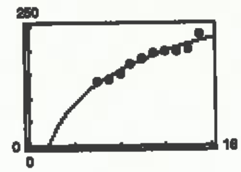
$I = -51.23 + 100.48 \ln t$

(b)

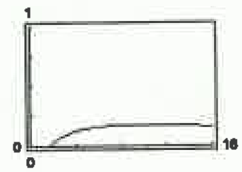


Rate of growth $= R'(t) \approx 32.1e^{0.056t}$

(c)



(d) $P(t) = \dfrac{I}{R}$



53. $\beta(I) = 10 \log_{10} \dfrac{I}{I_0}$, $I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \dfrac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10 \log_{10} \dfrac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10 \log_{10} \dfrac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10 \log_{10} \dfrac{10^{-4}}{10^{-16}} = 120$ decibels

54. $93 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-6.7 = \log_{10} I \Rightarrow I = 10^{-6.7}$

$80 = 10 \log_{10} \dfrac{I}{10^{-16}} = 10(\log_{10} I + 16)$

$-8 = \log_{10} I \Rightarrow I = 10^{-8}$

Percentage decrease: $\left(\dfrac{10^{-6.7} - 10^{-8}}{10^{-6.7}}\right)(100) \approx 95\%$

55. $R = \dfrac{\ln I - 0}{\ln 10}$, $I = e^{R \ln 10} = 10^R$

(a) $8.3 = \dfrac{\ln I - 0}{\ln 10}$

$I = 10^{8.3} \approx 199{,}526{,}231.5$

(b) $2R = \dfrac{\ln I - 0}{\ln 10}$

$I = e^{2R \ln 10} = e^{2R \ln 10} = (e^{R \ln 10})^2 = (10^R)^2$

Increases by a factor of $e^{2R \ln 10}$ or 10^R.

(c) $\dfrac{dR}{dI} = \dfrac{1}{I \ln 10}$

56. Since $\dfrac{dy}{dt} = k(y - 90)$,

$$\int \frac{1}{y - 90}\,dy = \int k\,dt$$

$\ln(y - 90) = kt + C.$

When $t = 0$, $y = 1500$. Thus, $C = \ln 1410$. When $t = 1$,

$y = 1120$. Thus,

$k(1) = \ln 1030 - \ln 1410$

$k = \ln \dfrac{103}{141}.$

Thus, $y = e^{[\ln(103/141)t + \ln 1410]} + 90$

$= 1410e^{[\ln(103/141)]t} + 90.$

When $t = 5$, $y = 1410e^{5 \ln(103/141)} + 90 \approx 383.298°.$

57. Since $\dfrac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y - 20}\,dy = \int k\,dt$$

$\ln(y - 20) = kt + C$

$y = Ce^{kt} + 20.$

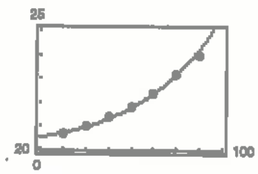
When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = (28/52) = (7/13)$, and $k = \ln(7/13)$. Thus, $y = 52e^{[\ln(7/13)]t} + 20$.

When $t = 5$, $y = 52e^{5 \ln(7/13)} + 20 \approx 22.35°.$

58. (a) $t = 1.5385(1.0296)^s \doteq 1.5385e^{0.02913s}$

(b)



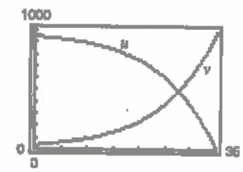
(c) $s = 55 \Rightarrow t \approx 7.6$ seconds

$$\frac{dt}{ds} = 0.0448e^{0.02913s}$$

At $s = 55, \dfrac{dt}{ds} = 0.22.$

59. (a) $u = 985.93 - \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$

$v = \left(985.93 - \dfrac{(120{,}000)(0.095)}{12}\right)\left(1 + \dfrac{0.095}{12}\right)^{12t}$



(b) The larger part goes for interest. The curves intersect when $t \approx 27.7$ years.

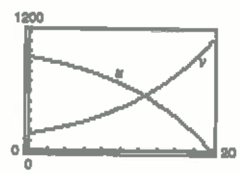
(c) The slopes are negatives of each other. Analytically,

$$u = 985.93 - v \quad \Rightarrow \frac{du}{dt} = -\frac{dv}{dt}$$

$u'(15) = -v'(15) \doteq -14.06.$

(d) $t = 12.7$ years

Again, the larger part goes for interest.



Section 5.7 Differential Equations: Separation of Variables

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

2. Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy} = \frac{-2xy}{2y^2 - (x^2 + y^2)} = \frac{-2xy}{y^2 - x^2} = \frac{2xy}{x^2 - y^2}$$

3. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check: $y' = -C_1 \sin x + C_2 \cos x$

$y'' = -C_1 \cos x - C_2 \sin x$

$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$

4. Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1e^{-x}\cos x + C_2e^{-x}\sin x$

Check:
$$y' = -(C_1 + C_2)e^{-x}\sin x + (-C_1 + C_2)e^{-x}\cos x$$
$$y'' = 2C_1e^{-x}\sin x - 2C_2e^{-x}\cos x$$
$$\begin{aligned} y'' + 2y' + 2y &= 2C_1e^{-x}\sin x - 2C_2e^{-x}\cos x + \\ &\quad 2(-(C_1 + C_2)e^{-x}\sin x + (-C_1 + C_2)e^{-x}\cos x) + 2(C_1e^{-x}\cos x + C_2e^{-x}\sin x) \\ &= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x}\sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x}\cos x = 0 \end{aligned}$$

5. $y = -\cos x\ln|\sec x + \tan x|$

$$\begin{aligned} y' &= (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x\ln|\sec x + \tan x| \\ &= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x\ln|\sec x + \tan x| \\ &= -1 + \sin x\ln|\sec x + \tan x| \end{aligned}$$

$$\begin{aligned} y'' &= (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x\ln|\sec x + \tan x| \\ &= (\sin x)(\sec x) + \cos x\ln|\sec x + \tan x| \end{aligned}$$

Substituting,
$$\begin{aligned} y'' + y &= (\sin x)(\sec x) + \cos x\ln|\sec x + \tan x| - \cos x\ln|\sec x + \tan x| \\ &= \tan x. \end{aligned}$$

6. $y = \frac{2}{3}(e^{-2x} + e^x)$

$y' = \frac{2}{3}(-2e^{-2x} + e^x)$

$y'' = \frac{2}{3}(4e^{-2x} + e^x)$

Substituting, $y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x)$

$= 2e^x.$

In Exercises 7–12, the differential equation is $y^{(4)} - 16y = 0$.

7.
$$\begin{aligned} y &= 3\cos x \\ y^{(4)} &= 3\cos x \\ y^{(4)} - 16y &= -45\cos x \neq 0, \quad \text{No.} \end{aligned}$$

8.
$$\begin{aligned} y &= 3\cos 2x \\ y^{(4)} &= 48\cos 2x \\ y^{(4)} - 16y &= 48\cos 2x - 48\cos 2x = 0, \quad \text{Yes.} \end{aligned}$$

9.
$$\begin{aligned} y &= e^{-2x} \\ y^{(4)} &= 16e^{-2x} \\ y^{(4)} - 16y &= 16e^{-2x} - 16e^{-2x} = 0, \quad \text{Yes.} \end{aligned}$$

10.
$$\begin{aligned} y &= 5\ln x \\ y^{(4)} &= -\frac{30}{x^4} \\ y^{(4)} - 16y &= -\frac{30}{x^4} - 80\ln x \neq 0, \quad \text{No.} \end{aligned}$$

11.
$$\begin{aligned} y &= C_1e^{2x} + C_2e^{-2x} + C_3\sin 2x + C_4\cos 2x \\ y^{(4)} &= 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3\sin 2x + 16C_4\cos 2x \\ y^{(4)} - 16y &= 0, \quad \text{Yes.} \end{aligned}$$

12.
$$y = 5e^{-2x} + 3\cos 2x$$
$$y^{(4)} = 80e^{-2x} + 48\cos 2x$$
$$y^{(4)} - 16y = (80e^{-2x} + 48\cos 2x) - (80e^{-2x} + 48\cos 2x) = 0, \quad \text{Yes.}$$

In 13–18, the differential equation is $xy' - 2y = x^3e^x$.

13. $y = x^2,\ y' = 2x$

$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$, No.

14. $y = x^2e^x,\ y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x$, Yes.

15. $y = x^2(2 + e^x),\ \ y' = x^2(e^x) + 2x(2 + e^x)$

$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x$, Yes.

16. $y = \sin x,\ y' = \cos x$

$xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x$, **No.**

17. $y = \ln x,\ y' = \dfrac{1}{x}$

$xy' - 2y = x\left(\dfrac{1}{x}\right) - 2\ln x \neq x^3e^x$, No.

18. $y = x^2e^x - 5x^2,\ y' = x^2e^x + 2xe^x - 10x$

$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x$, Yes.

19. $y = Ce^{kx}$

$$\frac{dy}{dx} = Cke^{kx}$$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

20. $y = A\sin \omega t$

$$\frac{d^2y}{dt^2} = -A\omega^2 \sin \omega t$$

Since $(d^2y/dt^2) + 16y = 0$, we have

$$-A\omega^2 \sin \omega t + 16A \sin \omega t = 0.$$

Thus, $\omega^2 = 16$ and $\omega = \pm 4$.

21. $y^2 = Cx^3$ passes through $(4, 4)$

$16 = C(64) \implies C = \frac{1}{4}$

Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

22. $2x^2 - y^2 = C$ passes through $(3, 4)$

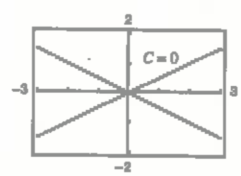
$2(9) - 16 = C \implies C = 2$

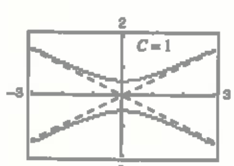
Particular solution: $2x^2 - y^2 = 2$

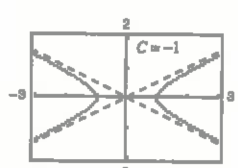
23. Differential equation: $4yy' - x = 0$

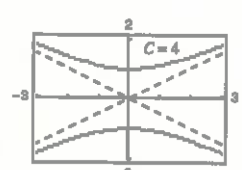
General solution: $4y^2 - x^2 = C$

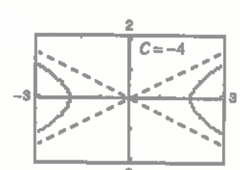
Particular solutions: $C = 0$, Two interesecting lines
$C = \pm 1,\ C = \pm 4$, Hyperbolas









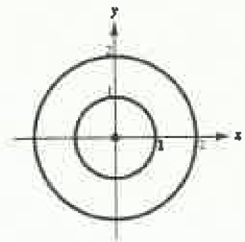


24. Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions: $C = 0$, Point

$C = 1, C = 4$, Circles



25. Differential equation: $y' + 2y = 0$

General Solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

26. Differential equation: $2x + 3yy' = 0$

General solution: $2x^2 + 3y^2 = C$

$4x + 6yy' = 0 \Rightarrow y' = \dfrac{-2x}{3y}$

$2x + 3yy' = 2x + 3y\left(\dfrac{-2x}{3y}\right) = 0$

Initial condition: $y(1) = 2, 2(1) + 3(4) = 14 = C$

Particular solution: $2x^2 + 3y^2 = 14$

27. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$

$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$

$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) +$

$9(C_1 \sin 3x + C_2 \cos 3x) = 0$

Initial conditions: $y\left(\dfrac{\pi}{6}\right) = 2, y'\left(\dfrac{\pi}{6}\right) = 1$

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right)$$

$$= -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

Particular solution: $y = 2 \sin 3x - \dfrac{1}{3} \cos 3x$

28. Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$y' = C_2\left(\dfrac{1}{x}\right), y'' = -C_2\left(\dfrac{1}{x^2}\right)$

$xy'' + y' = x\left(-C_2\dfrac{1}{x^2}\right) + C_2\dfrac{1}{x} = 0$

Initial conditions: $y(2) = 0,\ y'(2) = \dfrac{1}{2}$

$$0 = C_1 + C_2 \ln 2$$

$$y' = \frac{C_2}{x}$$

$$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1,\ C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln\dfrac{x}{2}$

29. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2, y'' = 6C_2x$

$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3) = 0$

Initial conditions: $y(2) = 0,\ y'(2) = 4$

$0 = 2C_1 + 8C_2$

$y' = C_1 + 3C_2x^2$

$4 = C_1 + 12C_2$

$$\left.\begin{array}{l} C_1 + 4C_2 = 0 \\ C_1 + 12C_2 = 4 \end{array}\right\} \quad C_2 = \frac{1}{2},\ C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

30. Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$$

$$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$$

$$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12(e^{2x/3})\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4(e^{2x/3})(C_1 + C_2x) = 0$$

Initial conditions: $y(0) = 4,\ y(3) = 0$

$0 = e^2(C_1 + 3C_2)$

$4 = (1)(C_1 + 0) \implies C_1 = 4$

$0 = e^2(4 + 3C_2) \implies C_2 = -\frac{4}{3}$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

31. $\frac{dy}{dx} = 3x^2$

$$y = \int 3x^2\,dx = x^3 + C$$

32. $\frac{dy}{dx} = \frac{x}{1 + x^2}$

$$y = \int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln(1 + x^2) + C$$

$(u = 1 + x^2,\ du = 2x\,dx)$

33. $\frac{dy}{dx} = \frac{x - 2}{x} = 1 - \frac{2}{x}$

$$y = \int\left[1 - \frac{2}{x}\right]dx$$

$$= x - 2\ln|x| + C = x - \ln x^2 + C$$

34. $\frac{dy}{dx} = x\cos x^2$

$$y = \int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C$$

$(u = x^2,\ du = 2x\,dx)$

35. $\dfrac{dy}{dx} = \sin 2x$

$y = \displaystyle\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

$(u = 2x,\ du = 2dx)$

36. $\dfrac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$y = \displaystyle\int (\sec^2 x - 1)\,dx = \tan x - x + C$

37. $\dfrac{dy}{dx} = x\sqrt{x-3}$ Let $u = \sqrt{x-3}$, then $x = u^2 + 3$ and $dx = 2u\,du$.

$y = \displaystyle\int x\sqrt{x-3}\,dx = \int (u^2+3)(u)(2u)du$

$= 2\displaystyle\int (u^4 + 3u^2)\,du = 2\left(\frac{u^5}{5} + u^3\right) + C = \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C$

38. $\dfrac{dy}{dx} = xe^{x^2}$

$y = \displaystyle\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$

$(u = x^2,\ du = 2x\,dx)$

39. $\dfrac{dy}{dx} = \dfrac{x}{y}$

$\displaystyle\int y\,dy = \int x\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + C_1$

$y^2 - x^2 = C$

40. $\dfrac{dy}{dx} = \dfrac{x^2+2}{3y^2}$

$\displaystyle\int 3y^2\,dy = \int (x^2+2)\,dx$

$y^3 = \dfrac{x^3}{3} + 2x + C$

41. $\dfrac{dr}{ds} = 0.05r$

$\displaystyle\int \frac{dr}{r} = \int 0.05\,ds$

$\ln|r| = 0.05s + C_1$

$r = e^{0.05s + C_1} = Ce^{0.05s}$

42. $\dfrac{dr}{ds} = 0.05s$

$\displaystyle\int dr = \int 0.05s\,ds$

$r = 0.025s^2 + C$

43. $(2+x)y' = 3y$

$\displaystyle\int \frac{dy}{y} = \int \frac{3}{2+x}\,dx$

$\ln y = 3\ln(2+x) + \ln C = \ln C(2+x)^3$

$y = C(x+2)^3$

44. $xy' = y$

$\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln y = \ln x + \ln C = \ln Cx$

$y = Cx$

45. $yy' = \sin x$

$\displaystyle\int y\,dy = \int \sin x\,dx$

$\dfrac{y^2}{2} = -\cos x + C_1$

$y^2 = -2\cos x + C$

46. $\sqrt{1-4x^2}\,\dfrac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$= -\frac{1}{8}\int (1-4x^2)^{-1/2}(-8x\,dx)$$

$$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$$

47. $y\ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$

$$\ln y = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

48. $y\dfrac{dy}{dx} = 2e^x$

$$\int y\,dy = \int 2e^x\,dx$$

$$\frac{y^2}{2} = 2e^x + C_1$$

$$y^2 = 4e^x + C$$

49. $yy' - e^x = 0$

$$\int y\,dy = \int e^x\,dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4,\ 16 = 2 + C,\ C = 14$

Particular solution: $y^2 = 2e^x + 14$

50. $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}\,dy = -\int x^{1/2}\,dx$$

$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4$, $(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

51. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1)\,dx$$

$$\ln y = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1,\ 1 = Ce^{-1/2},\ C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

52. $xyy' - \ln x = 0$

$$\int y\,dy = \int \frac{\ln x}{x}\,dx$$

$$\frac{y^2}{2} = \frac{\ln^2 x}{2} + C_1$$

$$y^2 = \ln^2 x + C$$

Initial condition: $y(1) = 0,\ 0 = \ln^2(1) + C = C$

Particular solution: $y^2 = \ln^2 x = (\ln x)^2$

53. $y(1+x^2)\dfrac{dy}{dx} = x(1+y^2)$

$$\frac{y}{1+y^2}\,dy = \frac{x}{1+x^2}\,dx$$

$$\frac{1}{2}\ln(1+y^2) = \frac{1}{2}\ln(1+x^2) + C_1$$

$$\ln(1+y^2) = \ln(1+x^2) + \ln C = \ln[C(1+x^2)]$$

$$1 + y^2 = C(1+x^2)$$

$$y(0) = \sqrt{3}:\ 1 + 3 = C \Rightarrow C = 4$$

$$1 + y^2 = 4(1+x^2)$$

$$y^2 = 3 + 4x^2$$

54. $y\sqrt{1-x^2}\dfrac{dy}{dx} = x\sqrt{1-y^2}$

$$\int (1-y^2)^{-1/2} y\,dy = \int (1-x^2)^{-1/2} x\,dx$$

$$-(1-y^2)^{1/2} = -(1-x^2)^{1/2} + C$$

$y(0) = 1$: $\quad 0 = -1 + C \Rightarrow C = 1$

$$\sqrt{1-y^2} = \sqrt{1-x^2} - 1$$

55. $\dfrac{du}{dv} = uv \sin v^2$

$$\int \frac{du}{u} = \int v \sin v^2\,dv$$

$$\ln u = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1$, $C = \dfrac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

56. $\dfrac{dr}{ds} = e^{r+s} = e^r e^s$

$$\int e^{-r}\,dr = \int e^s\,ds$$

$$-e^{-r} = e^s + C_1$$

$$e^s + e^{-r} = C$$

Initial condition: $r(1) = 0$, $C = e + 1$

Particular solution: $e^s + e^{-r} = e + 1$,

$$r = -\ln(e + 1 - e^s)$$

57. $dP - kP\,dt = 0$

$$\int \frac{dP}{P} = k\int dt$$

$$\ln P = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0$, $P_0 = Ce^0 = C$

Particular solution: $P = P_0 e^{kt}$

58. $dT + k(T - 70)\,dt = 0$

$$\int \frac{dT}{T-70} = -k\int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140$;
$140 - 70 = 70 = Ce^0 = C$

Particular solution: $T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

59. $\dfrac{dy}{dx} = \dfrac{-9x}{16y}$

$$\int 16y\,dy = -\int 9x\,dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition: $y(1) = 1$, $8 = -\dfrac{9}{2} + C$, $C = \dfrac{25}{2}$

Particular solution: $8y^2 = \dfrac{-9}{2}x^2 + \dfrac{25}{2}$,

$$16y^2 + 9x^2 = 25$$

60. $\dfrac{dy}{dx} = \dfrac{2y}{3x}$

$$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition: $y(8) = 2$, $2^3 = C(8^2)$, $C = \dfrac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \dfrac{1}{2}x^{2/3}$

61. $m = \dfrac{dy}{dx} = \dfrac{0 - y}{(x+2) - x} = -\dfrac{y}{2}$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$

$$\ln y = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

62. $m = \dfrac{dy}{dx} = \dfrac{y-0}{x-0} = \dfrac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

63. $f(x, y) = x^3 - 4xy^2 + y^3$

$$f(tx, ty) = t^3x^3 - 4t\,xt^2y^2 + t^3y^3$$
$$= t^3(x^3 - 4xy^2 + y^3)$$

Homogeneous of degree 3

64. $f(x, y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

$$f(tx, ty) = \frac{tx\,ty}{\sqrt{t^2x^2 + t^2y^2}}$$
$$= \frac{t^2xy}{t\sqrt{x^2+y^2}} = t\frac{xy}{\sqrt{x^2+y^2}}$$

Homogeneous of degree 1

65. $f(x, y) = 2\ln xy$

$$f(tx, ty) = 2\ln tx\,ty$$
$$= 2\ln t^2xy = 2(\ln t^2 + \ln xy)$$

Not homogeneous

66. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$$

Not homogeneous

67. $f(x, y) = 2\ln\dfrac{x}{y}$

$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous of degree 0

68. $f(x, y) = \tan\dfrac{y}{x}$

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x}$$

Homogeneous of degree 0

69. $y' = \dfrac{x+y}{2x},\ y = vx$

$$v + x\frac{dv}{dx} = \frac{x + vx}{2x}$$

$$x\frac{dv}{dx} = \frac{1+v}{2} - v$$

$$2\int\frac{dv}{1-v} = \int\frac{dx}{x}$$

$$-\ln(1-v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1-v^2)} = |Cx|$$

$$\frac{1}{[1-(y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x-y)^2} = |Cx| = C|x|$$

$$|x| = C(x-y)^2$$

70. $y' = \dfrac{(x^3+y^3)}{xy^2}$

$$xy^2\,dy = (x^3 + y^3)\,dx$$

$$y = vx, \quad dy = x\,dv + v\,dx$$

$$x(vx)^2(x\,dv + v\,dx) = (x^3 + (vx)^3)\,dx$$

$$x^4v^2\,dv + x^3v^3\,dx = x^3\,dx + v^3x^3\,dx$$

$$xv^2\,dv = dx$$

$$\int v^2\,dv = \int\frac{1}{x}\,dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3\ln|x| + C$$

$$y^3 = 3x^3\ln|x| + Cx^3$$

92. —CONTINUED—

(a) If (x, y) is the point of tangency on the $y = x - x^2$, then

$$1 - 2x = \frac{y-1}{x+1} = \frac{x - x^2 - 1}{x+1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4+8}}{2}\right) = -1 + \sqrt{3}$$

$$y = x - x^2 = 3\sqrt{3} - 5$$

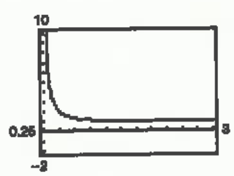
$$\text{Then } \frac{1-0}{-1-x_0} = \frac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \frac{6 - 3\sqrt{3}}{-\sqrt{3}}$$

$$\sqrt{3} = (1 + x_0)(6 - 3\sqrt{3})$$

$$= 6 - 3\sqrt{3} + x_0(6 - 3\sqrt{3})$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155$$

(c)



There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

(b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y-h}{x+1} = \frac{x - x^2 - h}{x+1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{\left(-2 \pm \sqrt{4 + 4(h+1)}\right)}{2}$$

$$= -1 + \sqrt{2+h}$$

$$y = x - x^2$$

$$= 3\sqrt{2+h} - h - 4$$

$$\text{Then, } \frac{h-0}{-1-x_0} = \frac{h - \left(3\sqrt{2+h} - h - 4\right)}{-1 - \left(-1 + \sqrt{2+h}\right)}$$

$$= \frac{2h + 4 - 3\sqrt{2+h}}{-\sqrt{2+h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2+h}}{2h + 4 - 3\sqrt{2+h}}$$

$$x_0 = \frac{h\sqrt{2+h}}{2h + 4 - 3\sqrt{2+h}} - 1$$

93. Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

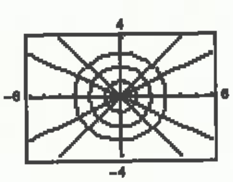
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$y = Kx$$



94. (a) Given family (hyperbolas): $2x^2 - y^2 = C$

$$4x - 2yy' = 0$$

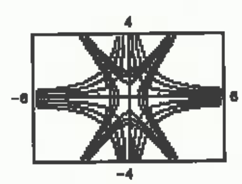
$$y' = \frac{2x}{y}$$

Orthogonal trajectory: $y' = -\frac{y}{2x}$

$$2\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$2 \ln y = -\ln x + \ln K$$

$$y^2 = \frac{K}{x}$$



95. Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$

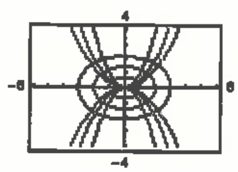
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



96. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

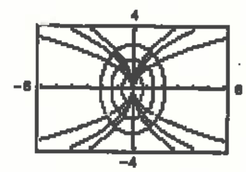
$$y' = \frac{C}{y} = \frac{y^2}{2x}\left(\frac{1}{y}\right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\frac{2x}{y}$

$$\int y\,dy = -\int 2x\,dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



97. Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

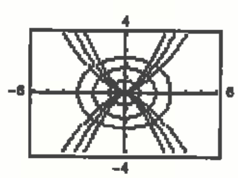
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



98. Given family (exponential functions): $y = Ce^x$

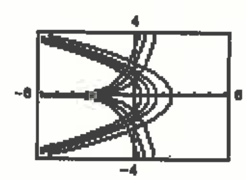
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y\,dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



99. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

100. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

101. False

$$f(tx, ty) = t^2x^2 + t^2xy + 2$$
$$\neq t^2 f(x, y)$$

102. True

$$x^2 + y^2 = 2Cy \qquad x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y} \qquad \frac{dy}{dx} = \frac{K - x}{y}$$

$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

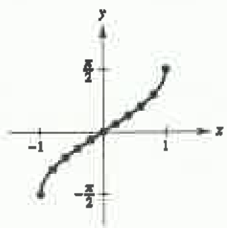
Section 5.8 Inverse Trigonometric Functions and Differentiation

1. $y = \arcsin x$

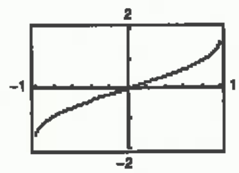
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571

(b)



(c)



(d) Symmetric about origin: $\arcsin(-x) = -\arcsin x$

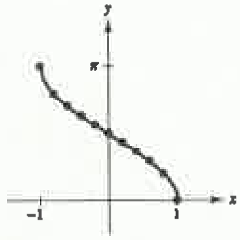
Intercept: $(0, 0)$

2. $y = \arccos x$

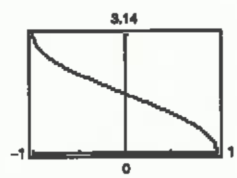
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.634	0

(b)



(c)



(d) Intercepts: $\left(0, \frac{\pi}{2}\right)$ and $(1, 0)$

No symmetry

3. False.

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

4. $\left(\_\_, \frac{\pi}{4}\right) = \left(1, \frac{\pi}{4}\right)$

$\left(\_\_, -\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{3}, -\frac{\pi}{6}\right)$

$\left(-\sqrt{3}, \_\_\right) = \left(-\sqrt{3}, -\frac{\pi}{3}\right)$

5. $\arcsin \frac{1}{2} = \frac{\pi}{6}$

6. $\arcsin 0 = 0$

7. $\arccos \frac{1}{2} = \frac{\pi}{3}$

8. $\arccos 0 = \frac{\pi}{2}$

9. $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

10. $\operatorname{arccot}(-1) = \frac{3\pi}{4}$

11. $\operatorname{arccsc}(-\sqrt{2}) = -\frac{\pi}{4}$

12. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

13. $\arccos(-0.8) \approx 2.50$

14. $\arcsin(-0.39) \approx -0.40$

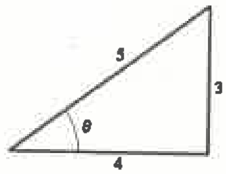
15. $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

16. $\arctan(-3) \approx -1.25$

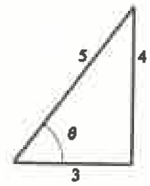
17. $\arctan 0 = 0$; π is not in the range of $y = \arctan x$.

18. $y = \arcsin x$ represents the inverse of the restricted sine function where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$.

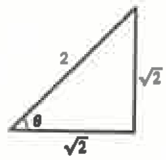
19. (a) $\sin\left(\arctan \frac{3}{4}\right) = \frac{3}{5}$



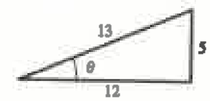
(b) $\sec\left(\arcsin \frac{4}{5}\right) = \frac{5}{3}$



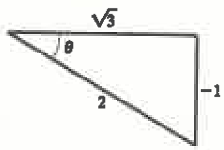
20. (a) $\tan\left(\arccos \dfrac{\sqrt{2}}{2}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1$



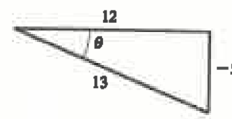
(b) $\cos\left(\arcsin \dfrac{5}{13}\right) = \dfrac{12}{13}$



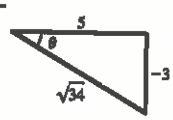
21. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$



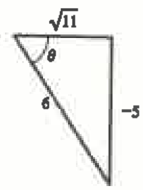
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$



22. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right] = \frac{\sqrt{34}}{5}$



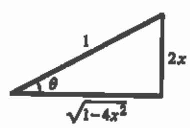
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right] = -\frac{5\sqrt{11}}{11}$



23. $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

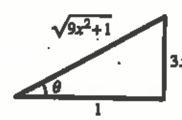
$y = \cos\theta = \sqrt{1 - 4x^2}$



24. $y = \sec(\arctan 3x)$

$\theta = \arctan 3x$

$y = \sec\theta = \sqrt{9x^2 + 1}$



25. $y = \sin(\operatorname{arcsec} x)$

$\theta = \operatorname{arcsec} x,\ 0 \le \theta \le \pi,\ \theta \ne \dfrac{\pi}{2}$

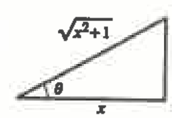
$y = \sin\theta = \dfrac{\sqrt{x^2 - 1}}{|x|}$

The absolute value bars on x are necessary because of the restriction $0 \le \theta \le \pi$, $\theta \ne \pi/2$, and $\sin\theta$ for this domain must always be nonnegative.

26. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

$y = \cos\theta = \dfrac{x}{\sqrt{x^2 + 1}}$



27. $y = \tan\left(\operatorname{arcsec} \dfrac{x}{3}\right)$

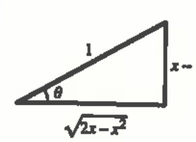
$\theta = \operatorname{arcsec} \dfrac{x}{3}$

$y = \tan\theta = \dfrac{\sqrt{x^2 - 9}}{3}$

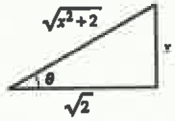
28. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

$y = \sec\theta = \dfrac{1}{\sqrt{2x - x^2}}$



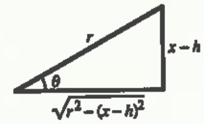
29. $y = \csc\left(\arctan \dfrac{x}{\sqrt{2}}\right)$



$\theta = \arctan \dfrac{x}{\sqrt{2}}$

$y = \csc \theta = \dfrac{\sqrt{x^2+2}}{x}$

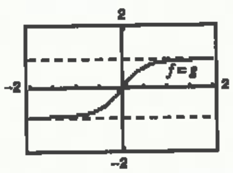
30. $y = \cos\left(\arcsin \dfrac{x-h}{r}\right)$



$\theta = \arcsin \dfrac{x-h}{r}$

$y = \cos \theta = \dfrac{\sqrt{r^2-(x-h)^2}}{r}$

31. $\sin(\arctan 2x) = \dfrac{2x}{\sqrt{1+4x^2}}$

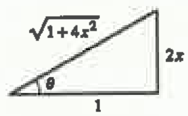


Asymptotes: $y = \pm 1$

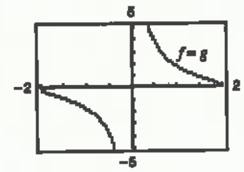
$\arctan 2x = \theta$

$\tan \theta = 2x$

$\sin \theta = \dfrac{2x}{\sqrt{1+4x^2}}$



32.

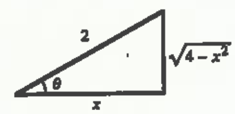


Asymptote: $x = 0$

$\arccos \dfrac{x}{2} = \theta$

$\cos \theta = \dfrac{x}{2}$

$\tan \theta = \dfrac{\sqrt{4-x^2}}{x}$



33. (a) $\operatorname{arccsc} x = \arcsin \dfrac{1}{x},\ |x| \geq 1$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \leq y < 0 \text{ and } 0 < y \leq \frac{\pi}{2},$$

$\csc y = x \Rightarrow \sin y = 1/x$. Thus, $y = \arcsin(1/x)$. Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

(b) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2},\ x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\tan y = \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x)\tan[\arctan(1/x)]}$$

$$= \frac{x + (1/x)}{1 - x(1/x)}$$

$$= \frac{x + (1/x)}{0} \text{ (which is undefined).}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

34. (a) $\arcsin(-x) = -\arcsin x,\ |x| \leq 1$.

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \Rightarrow x = -\sin y \Rightarrow x = \sin(-y).$$

Thus, $-y = \arcsin x \Rightarrow y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\arccos(-x) = \pi - \arccos x,\ |x| \leq 1$.

Let $y = \arccos(-x)$. Then,

$$-x = \cos y \Rightarrow x = -\cos y \Rightarrow x = \cos(\pi - y).$$

Thus, $\pi - y = \arccos x \Rightarrow y = \pi - \arccos x$. Therefore, $\arccos(-x) = \pi - \arccos x$.

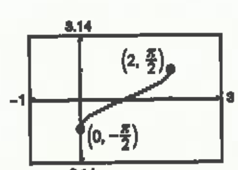
35. $f(x) = \arcsin(x-1)$

$x - 1 = \sin y$

$x = 1 + \sin y$

Domain: $[0, 2]$

Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$



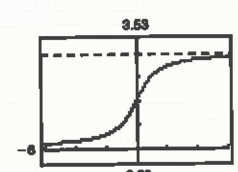
$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.

36. $f(x) = \arctan x + \dfrac{\pi}{2}$

$x = \tan\left(y - \dfrac{\pi}{2}\right)$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$



$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.

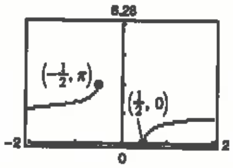
37. $f(x) = \text{arcsec } 2x$

$2x = \sec y$

$x = \frac{1}{2}\sec y$

Domain: $\left(-\infty, -\frac{1}{2}\right], \left[\frac{1}{2}, \infty\right)$

Range: $\left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$



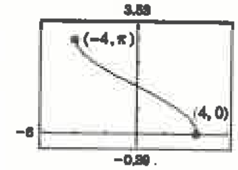
38. $f(x) = \arccos\left(\frac{x}{4}\right)$

$\frac{x}{4} = \cos y$

$x = 4\cos y$

Domain: $[-4, 4]$

Range: $[0, \pi]$



39. $\arcsin(3x - \pi) = \frac{1}{2}$

$3x - \pi = \sin\left(\frac{1}{2}\right)$

$x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right) + \pi\right] \approx 1.207$

40. $\arctan(2x) = -1$

$2x = \tan(-1)$

$x = \frac{1}{2}\tan(-1) \approx -0.779$

41. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

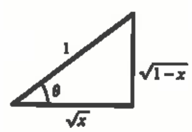
$\sqrt{2x} = \sin\left(\arccos\sqrt{x}\right)$

$\sqrt{2x} = \sqrt{1-x}, 0 \le x \le 1$

$2x = 1 - x$

$3x = 1$

$x = \frac{1}{3}$



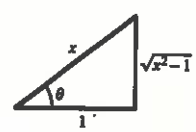
42. $\arccos x = \text{arcsec } x$

$x = \cos(\text{arcsec } x)$

$x = \frac{1}{x}$

$x^2 = 1$

$x = \pm 1$



43. $f(x) = 2\arcsin(x-1)$

$f'(x) = \frac{2}{\sqrt{1-(x-1)^2}} = \frac{2}{\sqrt{2x-x^2}}$

44. $f(t) = \arcsin t^2$

$f'(t) = \frac{2t}{\sqrt{1-t^4}}$

45. $g(x) = 3\arccos\frac{x}{2}$

$g'(x) = \frac{-3(1/2)}{\sqrt{1-(x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$

46. $f(x) = \text{arcsec } 2x$

$f'(x) = \frac{2}{|2x|\sqrt{4x^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}$

47. $f(x) = \arctan\frac{x}{a}$

$f'(x) = \frac{1/a}{1+(x^2/a^2)} = \frac{a}{a^2+x^2}$

48. $f(x) = \arctan\sqrt{x}$

$f'(x) = \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1+x)}$

49. $g(x) = \frac{\arcsin 3x}{x}$

$g'(x) = \frac{x\left(3/\sqrt{1-9x^2}\right) - \arcsin 3x}{x^2}$

$= \frac{3x - \sqrt{1-9x^2}\arcsin 3x}{x^2\sqrt{1-9x^2}}$

50. $h(x) = x\arctan x$

$h'(x) = \frac{x}{1+x^2} + \arctan x$

51. $h(t) = \sin(\arccos t) = \sqrt{1-t^2}$

$h'(t) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1-t^2}}$

52. $f(x) = \arcsin x + \arccos x = \frac{\pi}{2}$

$f'(x) = 0$

53. $y = \frac{1}{2}\left(\frac{1}{2}\ln\frac{x+1}{x-1} + \arctan x\right)$

$= \frac{1}{4}[\ln(x+1) - \ln(x-1)] + \frac{1}{2}\arctan x$

$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$

54. $y = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right)$

$\frac{dy}{dx} = \frac{1}{2}\left[x\left(\frac{-x}{\sqrt{1-x^2}}\right) + \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}}\right]$

$= \sqrt{1-x^2}$

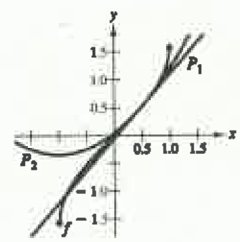
55. $y = x\arcsin x + \sqrt{1-x^2}$

$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1-x^2}}\right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$

56. $y = x\arctan 2x - \frac{1}{4}\ln(1+4x^2)$

$\frac{dy}{dx} = \frac{2x}{1+4x^2} + \arctan(2x) - \frac{1}{4}\left(\frac{8x}{1+4x^2}\right) = \arctan(2x)$

57. $f(x) = \arcsin x,\ a = \frac{1}{2}$



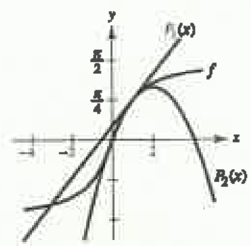
$f'(x) = \frac{1}{\sqrt{1-x^2}}$

$f''(x) = \frac{x}{(1-x^2)^{3/2}}$

$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$

$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$

58. $f(x) = \arctan x,\ a = 1$



$f'(x) = \frac{1}{1+x^2}$

$f''(x) = \frac{-2x}{(1+x^2)^2}$

$P_1(x) = f(1) + f'(1)(x-1) = \frac{\pi}{4} + \frac{1}{2}(x-1)$

$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$

59. $f(x) = \operatorname{arcsec} x - x$

$f'(x) = \frac{1}{|x|\sqrt{x^2-1}} - 1$

$= 0$ when $|x|\sqrt{x^2-1} = 1$.

$x^2(x^2-1) = 1$

$x^4 - x^2 - 1 = 0$ when $x^2 = \frac{1+\sqrt{5}}{2}$ or

$x = \pm\sqrt{\frac{1+\sqrt{5}}{2}} = \pm 1.272$

Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

60. $f(x) = \arcsin x - 2x$

$f'(x) = \frac{1}{\sqrt{1-x^2}} - 2 = 0$ when $\sqrt{1-x^2} = \frac{1}{2}$ or

$x = \pm\frac{\sqrt{3}}{2}$.

$f''(x) = \frac{x}{(1-x^2)^{3/2}}$

$f''\left(\frac{\sqrt{3}}{2}\right) > 0$

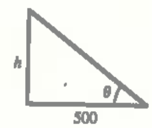
Relative minimum: $\left(\frac{\sqrt{3}}{2}, -0.68\right)$

$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$

Relative maximum: $\left(-\frac{\sqrt{3}}{2}, 0.68\right)$

61. (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec.



(b) $\tan\theta = \dfrac{h}{500} = \dfrac{-16t^2 + 256}{500}$

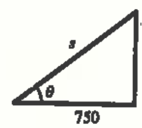
$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15{,}625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

62. $\cos\theta = \dfrac{750}{s}$



$$\theta = \arccos\left(\frac{750}{s}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{ds}\cdot\frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}}\left(\frac{-750}{s^2}\right)\frac{ds}{dt} = \frac{750}{s\sqrt{s^2 - 750^2}}\frac{ds}{dt}$$

63. $\tan(\arctan x + \arctan y) = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \dfrac{x + y}{1 - xy}, xy \neq 1$

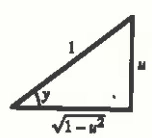
Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\frac{(1/2) + (1/3)}{1 - [(1/2)\cdot(1/3)]} = \arctan\frac{5/6}{1 - (1/6)} = \arctan\frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

64. (a) Let $y = \arcsin u$. Then

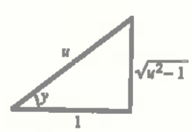


$\sin y = u$

$\cos y \cdot y' = u'$

$$\frac{dy}{dx} = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1 - u^2}}.$$

(c) Let $y = \operatorname{arcsec} u$. Then



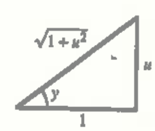
$\sec y = u$

$\sec y \tan y \dfrac{dy}{dx} = u'$

$$\frac{dy}{dx} = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2 - 1}}.$$

Note: The absolute value sign in the formula for the derivative of arcsec u is necessary because the inverse secant function has a positive slope at every value in its domain.

(b) Let $y = \arctan u$. Then

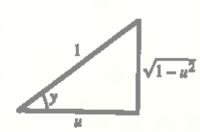


$\tan y = u$

$\sec^2 y \dfrac{dy}{dx} = u'$

$$\frac{dy}{dx} = \frac{u'}{\sec^2 y} = \frac{u'}{1 + u^2}.$$

(d) Let $y = \arccos u$. Then



$\cos y = u$

$-\sin y \dfrac{dy}{dx} = u'$

$$\frac{dy}{dx} = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1 - u^2}}.$$

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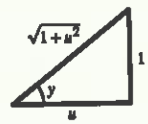
64. —CONTINUED—

(e) Let $y = \operatorname{arccot} u$. Then

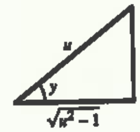


$$\cot y = u$$

$$-\csc^2 y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}.$$

(f) Let $y = \operatorname{arccsc} u$. Then



$$\csc y = u$$

$$-\csc y \cot y \frac{dy}{dx} = u'$$

$$\frac{dy}{dx} = \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Note: The absolute value sign in the formula for the derivative of arccsc u is necessary because the inverse cosecant function has a negative slope at every value in its domain.

65. $f(x) = kx + \sin x$

$f'(x) = k + \cos x \geq 0$ for $k \geq 1$

$f'(x) = k + \cos x \leq 0$ for $k \leq -1$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

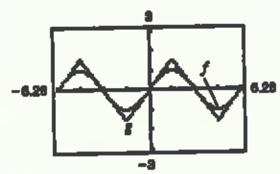
66. $f(x) = \sin x$

$g(x) = \arcsin(\sin x)$

(a) The range of $y = \arcsin x$ is $-\pi/2 \leq y \leq \pi/2$.

(b) Maximum: $\pi/2$

Minimum: $-\pi/2$



67. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} > 0 \text{ for all } x.$$

68. False

The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

69. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1+\tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

70. False

$$\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$$

Section 5.9 Inverse Trigonometric Functions and Integration

1. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\int_0^{1/6} \frac{1}{\sqrt{1-(3x)^2}}(3)\,dx = \left[\frac{1}{3}\arcsin(3x)\right]_0^{1/6} = \frac{\pi}{18}$$

2. $$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$$

3. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1+(2x)^2}\,dx = \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

4. $$\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx = \left[\frac{1}{3}\arctan\frac{x}{3}\right]_{\sqrt{3}}^3 = \frac{\pi}{36}$$

5. $$\int \frac{1}{x\sqrt{4x^2-1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2-1}}\,dx = \operatorname{arcsec}|2x| + C$$

6. $$\int \frac{1}{4+(x-1)^2}\,dx = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$$

7. $$\int \frac{x^3}{x^2+1}\,dx = \int\left[x - \frac{x}{x^2+1}\right]dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C \quad \text{(Use long division.)}$$

8. $$\int \frac{x^4-1}{x^2+1}\,dx = \int (x^2-1)\,dx = \frac{1}{3}x^3 - x + C$$

9. $$\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx = \arcsin(x+1) + C$$

10. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{t^4+16}\,dt = \frac{1}{2}\int \frac{1}{(4)^2+(t^2)^2}(2t)\,dt = \frac{1}{8}\arctan\frac{t^2}{4} + C$$

11. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{\sqrt{1-t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1-(t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin(t^2) + C$$

12. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{1}{x\sqrt{x^4-4}}\,dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{(x^2)^2-2^2}}(2x)\,dx$$
$$= \frac{1}{4}\operatorname{arcsec}\frac{x^2}{2} + C$$

13. Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

14. Let $u = \arccos x$, $du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx = -\int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}}\,dx$$
$$= \left[-\frac{1}{2}\arccos^2 x\right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925$$

15. Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_{-1/2}^0 (1-x^2)^{-1/2}(-2x)\,dx$$
$$= \left[-\sqrt{1-x^2}\right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2}$$
$$\approx -0.134$$

16. Let $u = 1 + x^2$, $du = 2x\,dx$.

$$\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{-\sqrt{3}}^{0} \frac{1}{1+x^2}(2x)\,dx = \left[\frac{1}{2}\ln(1+x^2)\right]_{-\sqrt{3}}^{0} = -\ln 2$$

17. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4+(e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$$

18. $$\int_1^2 \frac{1}{3+(x-2)^2}\,dx = \int_1^2 \frac{1}{(\sqrt{3})^2+(x-2)^2}\,dx = \left[\frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right)\right]_1^2 = \frac{\sqrt{3}\pi}{18}$$

19. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_{\pi/2}^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx = \left[-\arctan(\cos x)\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

20. Let $u = \sqrt{x}$, $du = \dfrac{1}{2\sqrt{x}}dx$

$$\int \frac{1}{\sqrt{x}(1+x)}\,dx = 2\int \frac{1}{1+u^2}\,du = 2\arctan(u) + C = 2\arctan(\sqrt{x}) + C$$

21. $$\int_0^2 \frac{1}{x^2-2x+2}\,dx = \int_0^2 \frac{1}{1+(x-1)^2}\,dx = \left[\arctan(x-1)\right]_0^2 = \frac{\pi}{2}$$

22. $$\int_{-3}^{-1} \frac{1}{x^2+6x+13}\,dx = \int_{-3}^{-1} \frac{1}{(x+3)^2+4}\,dx = \left[\frac{1}{2}\arctan\frac{x+3}{2}\right]_{-3}^{-1} = \frac{\pi}{8}$$

23. $$\int \frac{2x}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - 6\int \frac{1}{4+(x+3)^2}\,dx$$

$$= \ln|x^2+6x+13| - 3\arctan\left(\frac{x+3}{2}\right) + C$$

24. $$\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx - 7\int \frac{1}{1+(x+1)^2}\,dx = \ln|x^2+2x+2| - 7\arctan(x+1) + C$$

25. $$\int \frac{1}{\sqrt{-x^2-4x}}\,dx = \int \frac{1}{\sqrt{4-(x+2)^2}}\,dx = \arcsin\left(\frac{x+2}{2}\right) + C$$

26. $$\int \frac{1}{\sqrt{-x^2+2x}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \arcsin(x-1) + C$$

27. Let $u = -x^2 - 4x$, $du = (-2x-4)\,dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx = -\frac{1}{2}\int (-x^2-4x)^{-1/2}(-2x-4)\,dx = -\sqrt{-x^2-4x} + C$$

28. Let $u = x^2 - 2x$, $du = (2x-2)\,dx$.

$$\int \frac{x-1}{\sqrt{x^2-2x}}\,dx = \frac{1}{2}\int (x^2-2x)^{-1/2}(2x-2)\,dx = \sqrt{x^2-2x} + C$$

29. $$\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx = \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}}\,dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}}\,dx = -\int_2^3 (4x-x^2)^{-1/2}(4-2x)\,dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}}\,dx$$

$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

30. $$\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}}\,dx = \operatorname{arcsec}|x-1| + C$$

31. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2+1)^2 + 1}\,dx$$
$$= \frac{1}{2}\arctan(x^2 + 1) + C$$

32. Let $u = x^2 - 4$, $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}}\,dx$$
$$= \frac{1}{2}\arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

33. (a) $\displaystyle\int \frac{1}{\sqrt{1 - x^2}}\,dx = \arcsin x + C, u = x$

(b) $\displaystyle\int \frac{x}{\sqrt{1 - x^2}}\,dx = -\sqrt{1 - x^2} + C, u = 1 - x^2$

(c) $\displaystyle\int \frac{1}{x\sqrt{1 - x^2}}\,dx$ cannot be evaluated using the basic integration rules.

34. (a) $\displaystyle\int e^{x^2}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\displaystyle\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C, u = x^2$

(c) $\displaystyle\int \frac{1}{x^2}e^{1/x}\,dx = -e^{1/x} + C, u = \frac{1}{x}$

35. **(a)** $\displaystyle\int \sqrt{x - 1}\,dx = \frac{2}{3}(x - 1)^{3/2} + C, u = x - 1$

(b) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int x\sqrt{x - 1}\,dx = \int (u^2 + 1)(u)(2u)\,du = 2\int (u^4 + u^2)\,du = 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C$$
$$= \frac{2}{15}u^3(3u^2 + 5) + C = \frac{2}{15}(x - 1)^{3/2}[3(x - 1) + 5] + C = \frac{2}{15}(x - 1)^{3/2}(3x + 2) + C$$

(c) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x - 1}}\,dx = \int \frac{u^2 + 1}{u}(2u)\,du = 2\int (u^2 + 1)\,du = 2\left(\frac{u^3}{3} + u\right) + C = \frac{2}{3}u(u^2 + 3) + C = \frac{2}{3}\sqrt{x - 1}(x + 2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

36. (a) $\displaystyle\int \frac{1}{1 + x^4}\,dx$ cannot be evaluated using the basic integration rules.

(b) $\displaystyle\int \frac{x}{1 + x^4}\,dx = \frac{1}{2}\int \frac{2x}{1 + (x^2)^2}\,dx = \frac{1}{2}\arctan(x^2) + C, u = x^2$

(c) $\displaystyle\int \frac{x^3}{1 + x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1 + x^4}\,dx = \frac{1}{4}\ln(1 + x^4) + C, u = 1 + x^4$

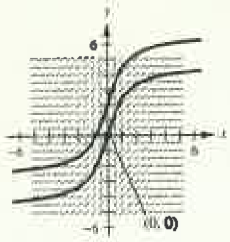
37. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2 + 3} = dt$.

$$\int \sqrt{e^t - 3}\,dt = \int \frac{2u^2}{u^2 + 3}\,du = \int 2\,du - \int 6\frac{1}{u^2 + 3}\,du$$
$$= 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t - 3}{3}} + C$$

38. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u\,du = dx$

$$\int \frac{\sqrt{x - 2}}{x + 1}\,dx = \int \frac{2u^2}{u^2 + 3}\,du = \int \frac{2u^2 + 6 - 6}{u^2 + 3}\,du = 2\int du - 6\int \frac{1}{u^2 + 3}\,du$$
$$= 2u - \frac{6}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{x - 2} - 2\sqrt{3}\arctan\sqrt{\frac{x - 2}{3}} + C$$

39. (a)



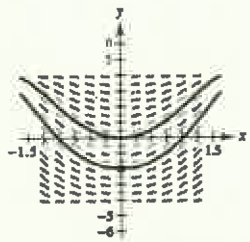
(b) $\dfrac{dy}{dx} = \dfrac{3}{1+x^2}, (0, 0)$

$$y = 3\int \frac{dx}{1+x^2} = 3 \arctan x + C$$

$$(0, 0): 0 = 3\arctan(0) + C \Rightarrow C = 0$$

$$y = 3 \arctan x$$

40. (a)



(b) $\dfrac{dy}{dx} = x\sqrt{16 - y^2}, (0, -2)$

$$\frac{dy}{\sqrt{16-y^2}} = x\,dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} + C$$

$$(0, -2): \arcsin\left(-\frac{2}{4}\right) = C \Rightarrow C = -\frac{\pi}{6}$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} - \frac{\pi}{6}$$

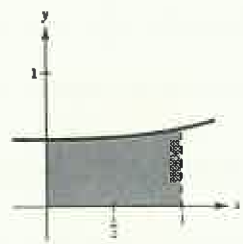
$$\frac{y}{4} = \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

$$y = 4\sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

41. $A = \displaystyle\int_1^3 \frac{1}{x^2 - 2x + 1 + 4}\,dx = \int_1^3 \frac{1}{(x-1)^2 + 2^2}\,dx$

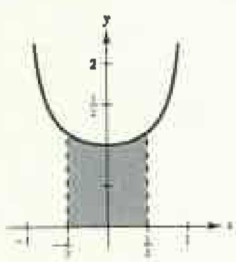
$$= \left[\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)\right]_1^3 = \frac{1}{2}\arctan(1) = \frac{\pi}{8} \approx 0.3927$$

42. $A = \displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}$

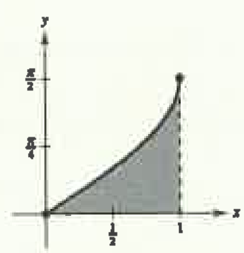


43. Area $\approx (1)(1) = 1$

Matches (c)



44. $\displaystyle\int_0^1 \arcsin x\,dx \approx 0.571$



45. (a) $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \Big[4\arctan x\Big]_0^1 = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4\int_0^1 \frac{4}{1+x^2}\,dx \approx 4\left(\frac{1}{36}\right)\left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2}\right] \approx 3.1415918$$

(c) 3.1415927

46. $F(x) = \displaystyle\frac{1}{2}\int_x^{x+2} \frac{2}{t^2+1}\,dt$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x+2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

(b) $F(x) = \Big[\arctan t\Big]_x^{x+2} = \arctan(x+2) - \arctan x$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2) - (x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

47. $\displaystyle\int \frac{1}{\sqrt{6x - x^2}}\,dx$

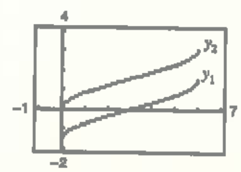
(a) $6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$

$$\int \frac{1}{\sqrt{6x-x^2}}\,dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

(b) $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$$\int \frac{1}{\sqrt{6u^2 - u^4}}(2u\,du) = \int \frac{2}{\sqrt{6-u^2}}\,du = 2\arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2\arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

(c)



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

48. (a) $\displaystyle\frac{d}{dx}\left[\arcsin\left(\frac{u}{a}\right) + C\right] = \frac{1}{\sqrt{1-(u^2/a^2)}}\left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$

Thus, $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$

(b) $\displaystyle\frac{d}{dx}\left[\frac{1}{a}\arctan\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{1+(u/a)^2}\right] = \frac{1}{a^2}\left[\frac{u'}{(a^2+u^2)/a^2}\right] = \frac{u'}{a^2+u^2}$

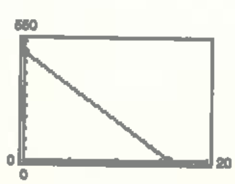
Thus, $\displaystyle\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C.$

(c) Assume $u > 0$.

$\displaystyle\frac{d}{dx}\left[\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}}\right] = \frac{1}{a}\left[\frac{u'}{u\sqrt{(u^2-a^2)/a^2}}\right] = \frac{u'}{u\sqrt{u^2-a^2}}$. The case $u < 0$ is handled in a similar manner.

Thus, $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}}\,dx = \frac{1}{a}\operatorname{arsec}\frac{|u|}{a} + C.$

49. (a) $v(t) = -32t + 500$



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49. —CONTINUED—

(b) $s(t) = \int v(t)\,dt = \int (-32t + 500)\,dt$

$= -16t^2 + 500t + C$

$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$

$s(t) = -16t^2 + 500t$

When the object reaches its maximum height, $v(t) = 0$.

$v(t) = -32t + 500 = 0$

$-32t = -500$

$t = 15.625$

$s(15.625) = -16(15.625)^2 + 500(15.625)$

$= 3906.25$ ft (Maximum height)

(c) $\displaystyle\int \frac{1}{32 + kv^2}\,dv = -\int dt$

$$\frac{1}{\sqrt{32k}}\arctan\left(\sqrt{\frac{k}{32}}v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}}v\right) = -\sqrt{32k}t + C$$

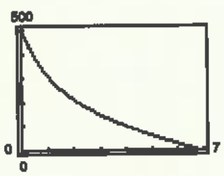
$$\sqrt{\frac{k}{32}}v = \tan\left(C - \sqrt{32k}\,t\right)$$

$$v = \sqrt{\frac{32}{k}}\tan\left(C - \sqrt{32k}\,t\right)$$

When $t = 0$, $v = 500$, $C = \arctan\left(500\sqrt{k/32}\right)$, and we have

$$v(t) = \sqrt{\frac{32}{k}}\tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}\,t\right].$$

(d) When $k = 0.001$, $v(t) = \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]$.



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

(e) $h = \displaystyle\int_0^{6.86} \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]dt$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

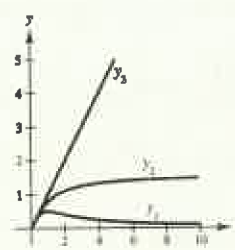
(f) Air resistance lowers the maximum height.

50. Let $f(x) = \arctan x - \dfrac{x}{1 + x^2}$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1 - x^2}{(1 + x^2)^2} = \frac{2x^2}{(1 + x^2)} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$, $\arctan x - \dfrac{x}{1 + x^2} > 0$ for $x > 0$. Thus,

$$\arctan x > \frac{x}{1 + x^2}.$$



Let $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$.

Therefore,

$$\frac{x}{1 + x^2} < \arctan x < x.$$