APCalc Date:________________
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Unit 6: Applications of Integration
6.2 Differential Equations – Growth & Decay

Law of Exponential Growth & Decay

If the rate of change of y with respect to time t is directly proportional to y (ie. the rate at which the quantity is changing at a given time depends on how much of the quantity there is at that time) determine an expression for y as a function of t.

$$\frac{dy}{dt} = ky$$

rate of change of y with respect to time t directly proportional to y (*k is a constant)

*solve the DE with separation of variables

$$\frac{dy}{y} = k\,dt$$

$$\int \frac{dy}{y} = \int k\ dt$$

$$\ln|y| = kt + c$$

$$|y| = e^{kt+c}$$

$$y = \pm e^{c} \cdot e^{kt}$$

$$\boxed{y = Ce^{kt}}$$

This does not need to be repeated every time. We have proven a general formula to deal with growth & decay questions which match the given parameters (the rate of change of y with respect to time t is directly proportional to).

* $C =$ initial quantity at $t = 0$
* $k =$ constant of growth/decay: $k > 0$ - growth
  $k < 0$ - decay

ex. The rate of change of y with respect to time t is directly proportional to y. If $y = 2$ when $t = 0$ and $y = 4$ when $t = 2$, find y when $t = 3$.

*use given conditions to determine C & k:

$$y = Ce^{kt}$$

$$y = 2e^{kt}$$

$$(4) = 2e^{k(2)}$$

$$2 = e^{2k}$$

$$2k = \ln 2$$

$$k = \frac{\ln 2}{2} = \ln\sqrt{2}$$

$$y = 2e^{\mathrm{t}\ln\sqrt{2}}$$

$$y = 2e^{(3)\ln\sqrt{2}}$$

$$\boxed{\cong 5.657}$$

ex. The rate of decay of plutonium is proportional to the amount present at time t. If it has a half-life of 24, 360 years, how long will it take for 10 g to decay to 1 g?

$$y = Ce^{kt}$$
$$y = 10e^{kt}$$
$$5 = 10e^{k(24360)}$$
$$0.5 = e^{24360k}$$
$$24360k = \ln 0.5$$
$$k = \frac{\ln 0.5}{24360}$$

*note: for any half-life scenario: $k = \frac{\ln 0.5}{\text{half-life}}$

$$y = 10e^{\frac{\ln 0.5}{24360}t}$$
$$1 = 10e^{\frac{\ln 0.5}{24360}t}$$
$$0.1 = e^{\frac{\ln 0.5}{24360}t}$$
$$\frac{\ln 0.5}{24360}t = \ln 0.1$$
$$t = \frac{24360 \ln 0.1}{\ln 0.5}$$
$$\boxed{t \cong 80{,}923\ years}$$

ex. The rate of growth of a population of fruit flies is directly proportional to the population at time t. There were 100 flies after 2 days and 300 after 4 days. How many flies in the original population?

*create a system of equations:

$$y = Ce^{kt} \qquad\qquad y = Ce^{kt}$$
$$100 = Ce^{2k} \qquad\qquad 300 = Ce^{4k}$$

*divide:

$$\frac{300}{100} = \frac{Ce^{4k}}{Ce^{2k}}$$
$$3 = e^{2k}$$
$$2k = \ln 3$$
$$k = \frac{\ln 3}{2} = \ln \sqrt{3}$$

*sub back into either equation to find C:

$$100 = Ce^{2\ln\sqrt{3}}$$
$$C = \frac{100}{e^{\ln 3}}$$
$$= \frac{100}{3}$$
$$\boxed{\cong 33\ flies}$$

Practice Questions

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