Unit 6: Applications of Integration
6.1 Differential Equations – Separation of Variables

*recall: solving a differential equation (DE) means to find the antiderivative of a given function

ex. Find the general solution of $\frac{dy}{dx} = x^2$

$$dy = x^2\,dx$$
$$y = \int x^2\,dx$$
$$\boxed{y = \frac{x^3}{3} + c}$$

We can solve more complex DE's by *separation of variables* – splitting the two variables so they are separated by the equal sign.

ex. Find the general solution of $(x^2+4)\frac{dy}{dx} = xy$

*separate so that all x terms are on right side with dx and all y terms are on left side with dy. Then integrate both sides with respect to that side's variable.

$$\frac{1}{y}dy = \frac{x}{x^2+4}dx$$
$$\int \frac{1}{y}dy = \int \frac{x}{x^2+4}dx$$

let $u = x^2 + 4$
then $\frac{du}{dx} = 2x$
and $dx = \frac{du}{2x}$

$$\int \frac{1}{y}dy = \int \frac{x}{u}\frac{du}{2x}$$
$$\int \frac{1}{y}dy = \frac{1}{2}\int \frac{1}{u}du$$
$$\ln|y| = \frac{1}{2}\ln(x^2+4) + c$$

*solve for y: rewrite in exponential form

$$|y| = e^{\ln\left(\sqrt{x^2+4}\right)+c}$$
$$|y| = e^{\ln\left(\sqrt{x^2+4}\right)} \cdot e^c$$
$$y = \pm e^c\sqrt{x^2+4}$$
$$\boxed{y = C\sqrt{x^2+4}}$$

ex. Find the particular solution of $yy' = cosx$ if $y(0) = 2$

$$y\frac{dy}{dx} = \cos x$$
$$y\,dy = \cos x\,dx$$
$$\int y\ dy = \int \cos x\,dx$$
$$\frac{y^2}{2} = \sin x + c$$
$$y^2 = 2\sin x + C$$
$$y = \pm\sqrt{2\sin x + C}$$
$$(2) = \pm\sqrt{2\sin(0) + C}$$
$$4 = 2(0) + C$$
$$C = 4$$
$$\boxed{y = +\sqrt{2\sin x + 4}}$$

+ only because for the specific point given $y = +$

ex. Given the DE $y'' - y = 0$ determine which of the following functions are solutions:

a) $y = \sin x$

b) $y = Ce^x$

*check by substitution

a)

$y' = \cos x$
$y'' = -\sin x$

$y'' - y = 0$
$-\sin x - \sin x \neq 0$

$y = \sin x$ is NOT a solution

b)

$y' = Ce^x$
$y'' = Ce^x$

$y'' - y = 0$
$Ce^x - Ce^x = 0$

$y = Ce^x$ IS a solution

Practice Questions

Pg. 374/ #3, 9, 13, 25, 31, 33, 39, 40, 43, 45, 47 – 49, 51 – 53, 55, 56