APCalc
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Date:________________

Unit 4: Integration
4.6 Mean Value Theorem & The 2nd FTC

Theorem

If f is continuous on $[a, b]$, then there exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Geometric interpretation: there is a rectangle with height $f(c)$ and width $(b-a)$ which has area equal to the actual area under the curve. [applet]

The value of $f(c)$ in the theorem is called the ***average value*** of f on $[a, b]$.
*not the same as average rate of change of f

Definition

The average value of f on $[a, b]$ is

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

ex. Find the average value of $f(x) = 3x^2 - 2x$ on $[1, 4]$ and determine the value c in the interval where the average value occurs.

$$\begin{aligned}
\text{average value} = f(c) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{4-1}\int_1^4 3x^2 - 2x\ \,dx \\
&= \frac{1}{3}[x^3 - x^2]_1^4 \\
&= \frac{1}{3}[(64-16) - (1-1)] \\
&\boxed{= 16}
\end{aligned}$$

$$\therefore f(c) = 3c^2 - 2c = 16$$

$$3c^2 - 2c - 16 = 0$$

$$(c+2)(3c-8) = 0$$

$$\cancel{c = -2} \ or \ \boxed{c = \frac{8}{3}}$$

If the upper bound of a definite integral is a variable, instead of a number, then the definite integral will be a function.

ex.

$$\int_0^x \cos t \, dt$$

$$= [\sin t]_0^x$$

$$= \sin x - \sin 0$$

$$\boxed{= \sin x}$$

ex.

$$\int_2^x 2t \, dt$$

$$= [t^2]_2^x$$

$$\boxed{= x^2 - 4}$$

If we take the derivative of one of these integral functions note how the answer becomes the intgrand with a change of variable.

ex.

$$\frac{d}{dx}\int_0^x \cos t \, dt$$

$$= \frac{d}{dx}[\sin x]$$

$$\boxed{= \cos x}$$

ex.

$$\frac{d}{dx}\int_2^x 2t \, dt$$

$$= \frac{d}{dx}[x^2 - 4]$$

$$\boxed{= 2x}$$

*note: the value of the lower limit is irrelevant

The 2nd Fundamental Theorem of Calculus

If f is continuous on an open interval containing a value c, then for every x in the interval

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x) \quad \text{and} \quad \frac{d}{dx}\int_c^{g(x)} f(t)\,dt = f(g(x)) \cdot\ g'(x)$$

ex.

If $F(x) = \displaystyle\int_3^x 3t^2 + 2\,dt,$ find $F'(x)$

$$F'(x) = \frac{d}{dx}\int_3^x 3t^2 + 2\,dt$$

$$\boxed{= 3x^2 + 2}$$

ex. Evaluate

$$\frac{d}{dx}\int_{\pi/2}^{x^3} \cos t\,dt$$

$= \cos x^3 \cdot \frac{d}{dx}[x^3]$ *chain rule

$$\boxed{= 3x^2 \cos x^3}$$

note: The constant **must** be the lower limit of integration.

ex. Evaluate

$$\frac{d}{dx}\int_{3x^2}^{5} 3 - 2x\,dt$$

$$= \frac{d}{dx}\left[-\int_5^{3x^2} 3 - 2x\,dt\right]$$

$$= -\left[\left(3 - 2(3x^2)\right) \cdot 6x\right]$$

$$\boxed{= 36x^3 - 18x}$$

Practice Questions

Pg. 284/# 43, 44, 46 – 50, 56, 57, 60, 66 – 70, 73 – 84, 89 – 91