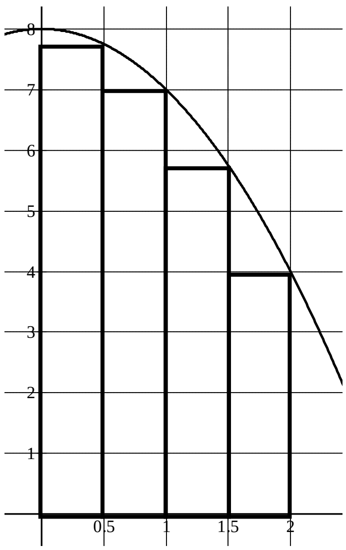
Unit 4: Integration
4.3 Riemann Sums

ex. Find a general expression for the lower and upper sums of $f(x) = -x^2 + 8$ on $[0, 2]$ using n rectangles.

Lowersum = right endpt rectangles



width of each rectangle:

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

right endpts occur at:

$$a + i\Delta x = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

height of i^{th} rectangle:

$$f\left(\frac{2i}{n}\right) = -\left(\frac{2i}{n}\right)^2 + 8 = -\frac{4i^2}{n^2} + 8$$

Area of i^{th} rectangle:

$$A = wh = \frac{2}{n}\left(-\frac{4i^2}{n^2} + 8\right)$$

Sum of n rectangles:

$$\sum_{i=1}^{n} \frac{2}{n}\left(-\frac{4i^2}{n^2} + 8\right)$$

$$= \frac{-8}{n}\left[\sum_{i=1}^{n} \frac{i^2}{n^2} - \sum_{i=1}^{n} 2\right]$$

$$= \frac{-8}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} 2\right]$$

$$= \frac{-8}{n}\left[\frac{1}{n^2}\left(\frac{n(2n^2 + 3n + 1)}{6}\right) - 2n\right]$$

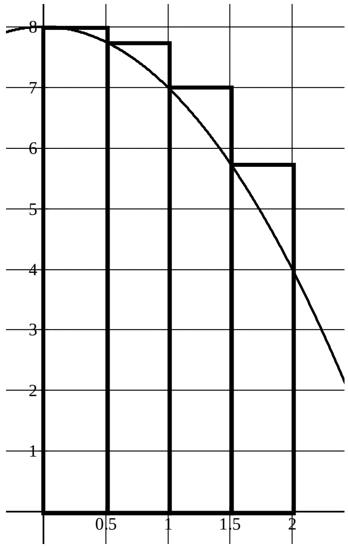
$$= -8\left[\frac{1}{n^2}\left(\frac{2n^2 + 3n + 1}{6}\right) - 2\right]$$

$$= -8\left[\frac{2n^2 + 3n + 1}{6n^2} - \frac{12n^2}{6n^2}\right]$$

$$= -4\left[\frac{-10n^2 + 3n + 1}{3n^2}\right]$$

$$\boxed{Lowersum = \frac{40n^2 - 12n - 4}{3n^2}}$$

Uppersum = left endpt rectangles



width of each rectangle:

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

left endpts occur at:

$$a + (i-1)\Delta x = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$$

height of i^{th} rectangle:

$$f\left(\frac{2(i-1)}{n}\right) = -\left(\frac{2(i-1)}{n}\right)^2 + 8 = -\frac{4(i^2-2i+1)}{n^2} + 8$$

Area of i^{th} rectangle:

$$A = wh = \frac{2}{n}\left(-\frac{4(i^2-2i+1)}{n^2} + 8\right)$$

Sum of n rectangles:

$$\sum_{i=1}^{n} \frac{2}{n}\left(-\frac{4(i^2-2i+1)}{n^2} + 8\right)$$

$$= \frac{-8}{n}\left[\sum_{i=1}^{n} \frac{i^2-2i+1}{n^2} - \sum_{i=1}^{n} 2\right]$$

$$= \frac{-8}{n}\left[\frac{1}{n^2}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right] - \sum_{i=1}^{n} 2\right]$$

$$= \frac{-8}{n}\left[\frac{1}{n^2}\left[\left(\frac{n(2n^2+3n+1)}{6}\right) - 2\left(\frac{n(n+1)}{2}\right) + n\right] - 2n\right]$$

$$= -8\left[\frac{1}{n^2}\left[\left(\frac{2n^2+3n+1}{6}\right) - (n+1) + 1\right] - 2\right]$$

$$= -8\left[\frac{1}{n^2}\left[\left(\frac{2n^2+3n+1}{6}\right) - n - 1 + 1\right] - 2\right]$$

$$= -8\left[\left[\left(\frac{2n^2+3n+1}{6n^2}\right) - \frac{1}{n}\right] - 2\right]$$

$$= -8\left[\frac{2n^2+3n+1}{6n^2} - \frac{6n}{6n^2} - \frac{12n^2}{6n^2}\right]$$

$$= -4\left[\frac{-10n^2-3n+1}{3n^2}\right]$$

$$\boxed{Uppersum = \frac{40n^2+12n-4}{3n^2}}$$

To find the *actual* area, take the limit as n approaches infinity:

Limit of lowersum:

$$\lim_{n\to\infty} \frac{40n^2 - 12n - 4}{3n^2} = \frac{40}{3}\ un^2$$

Limit of uppersum:

$$\lim_{n\to\infty} \frac{40n^2 + 12n - 4}{3n^2} = \frac{40}{3}\ un^2$$

This is NOT a coincidence, to find the actual area it does not matter which endpts are used. Thus, use the **right endpts** as they are easier.

Riemann Sum Approximation

The area under $f(x)$ on $[a, b]$ using n rectangles.

Width of each rectangle $\Delta x = \frac{b-a}{n}$

Left endpts $a + (i-1)\Delta x$

Right endpts $a + i\Delta x$

$$Lowersum = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$$

$$Uppersum = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$$

[where m_i and M_i are the left or right endpts depending which give the desired sum]

Note: $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n)$

Definition of Area of a Region

Let f be continuous and non-negative on $[a, b]$, then the area of the region bounded by f, the x-axis, $x = a$, and $x = b$ is given by:

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

[where $\Delta x = \frac{b-a}{n}$ and c_i is an x-value in the i^{th} rectangle]

*We will let c_i be the right endpts.

ex. Find the area under $f(x) = x^2$ on $[2, 5]$

$\Delta x = \frac{5-2}{n} = \frac{3}{n}$ right endpts $= 2 + \frac{3i}{n}$ height $= \left(2 + \frac{3i}{n}\right)^2 = 4 + \frac{12i}{n} + \frac{9i^2}{n^2}$

Thus, the Riemann sum approximation is:

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{3}{n}\left(4 + \frac{12i}{n} + \frac{9i^2}{n^2}\right) \\
&= \frac{3}{n}\left[\sum_{i=1}^{n} 4 + \frac{12}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2\right] \\
&= \frac{3}{n}\left[4n + \frac{12}{n}\left(\frac{n(n+1)}{2}\right) + \frac{9}{n^2}\left(\frac{n(2n^2+3n+1)}{6}\right)\right] \\
&= 3\left[4 + \frac{6}{n}(n+1) + \frac{3}{n^2}\left(\frac{2n^2+3n+1}{2}\right)\right] \\
&= 3\left[\frac{8n^2}{2n^2} + \frac{12n^2+12n}{2n^2} + \frac{6n^2+9n+3}{2n^2}\right] \\
&= 3\left[\frac{26n^2+21n+3}{2n^2}\right] \\
&= \frac{78n^2+63n+9}{2n^2}
\end{aligned}$$

Then, the area will be:

$$\begin{aligned}
A &= \lim_{n\to\infty} \frac{78n^2+63n+9}{2n^2} \\
&= \frac{78}{2} \\
&\boxed{= 39\ un^2}
\end{aligned}$$

Practice Questions

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