

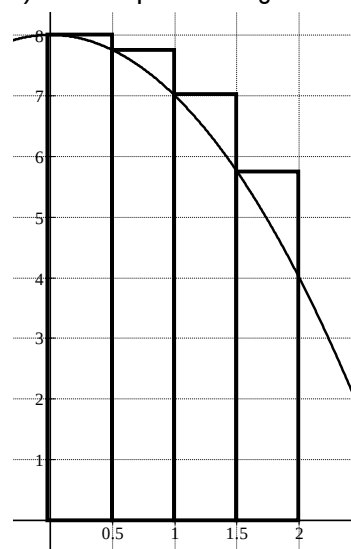
Unit 4: Integration
4.2 Area Approximation & Sigma Notation

Approximate the area under a curve using rectangles of equal width.

ex. Use four rectangles to approximate the area under $f(x) = -x^2 + 8$ on $[0, 2]$.

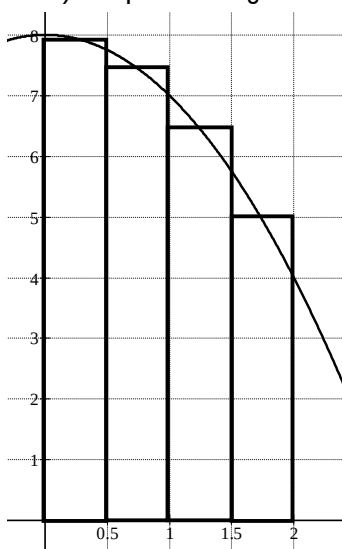
3 approaches

1) Left endpt. Rectangles



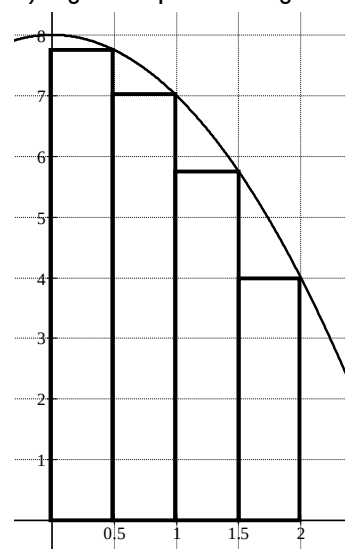
upper sum: area > actual

2) Midpt. Rectangles



midsum

3) Right endpt. Rectangles



lower sum: area < actual

$$\text{width of each rectangle: } w = \frac{2-0}{4} = 0.5$$

leftpts: $x = 0, 0.5, 1, 1.5$

midpts: $x = 0.25, 0.75, 1.25, 1.75$

rightpts: $x = 0.5, 1, 1.5, 2$

Area = sum of the rectangle areas (ie. $A = \sum \text{rectangle areas}$)

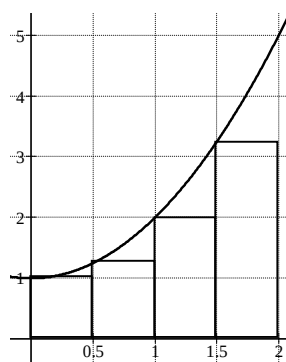
$$\begin{aligned} &= wh_1 + wh_2 + wh_3 + wh_4 \\ &= w(h_1 + h_2 + h_3 + h_4) \end{aligned}$$

$$\begin{aligned} A &= 0.5(f(0) + f(0.5) + f(1) + f(1.5)) & A &= 0.5(f(0.25) + f(0.75) + f(1.25) + f(1.75)) & A &= 0.5(f(0.5) + f(1) + f(1.5) + f(2)) \\ &= 0.5(8 + 7.75 + 7 + 5.75) & &= 0.5(7.9375 + 7.4375 + 6.4375 + 4.9375) & &= 0.5(7.75 + 7 + 5.75 + 4) \\ &= 14.25 \text{ un}^2 & &= 13.375 \text{ un}^2 & &= 12.25 \text{ un}^2 \end{aligned}$$

Which is best? Probably the midsum (but also the most tedious to calculate).

How can we improve the approximation? Increase the number of rectangles. [\[applet1\]](#)[\[applet2\]](#)

*note: left endpts. does not always mean the uppersum (depends on if function is inc/dec)



ex. left endpts. = lowersum

Larger number of rectangles (thus leading to narrower width rectangles) will improve the approximation but becomes more tedious to calculate. Thus, we use sigma notation and summation formulas.

Recall, we represent sums using the Greek letter Sigma Σ :

ex. "The sum from k equals one to five of two k minus three."

$$\sum_{k=1}^5 2k - 3$$

upper limit $\rightarrow 5$

term generator $\rightarrow t_n$ formula, generates the terms of the series

index of summation $\rightarrow k=1$

lower limit

*same variable in term generator

* to find t_1 let the index variable = lower limit
to find t_2 let the index variable = lower limit + 1
to find t_3 let the index variable = lower limit + 2
etc...
to find last term let the index variable = upper limit

ex.

$$\sum_{i=1}^5 i = 1 + 2 + 3 + 4 + 5$$

$$\sum_{j=3}^6 j^2 = 9 + 16 + 25 + 36$$

Summation Properties

$$\sum_{i=1}^n k a_i = k \sum_{i=1}^n a_i \quad \text{where } k \text{ is a constant, and } a_i \text{ is the term generator}$$

ex.

$$\sum_{i=1}^3 2i^3 = 2 \sum_{i=1}^3 i^3 = 2[1 + 8 + 27]$$

$$\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$$

ex.

$$\sum_{b=2}^3 (b + b^2) = \sum_{b=2}^3 b + \sum_{b=2}^3 b^2 = [2 + 3] + [4 + 9]$$

Summation Formulas

$$\sum_{i=1}^n c = cn \text{ where } c \text{ is a constant}$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n(2n^2 + 3n + 1)}{6}$$

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

ex. evaluate for $n = 10$ and $n = 100$:

$$\begin{aligned} & \sum_{i=1}^n \frac{i+1}{n^2} \\ &= \frac{1}{n^2} \sum_{i=1}^n (i+1) \\ &= \frac{1}{n^2} \left[\sum_{i=1}^n i + \sum_{i=1}^n 1 \right] \\ &= \frac{1}{n^2} \left[\frac{n(n+1)}{2} + n \right] \\ &= \frac{1}{n} \left[\frac{n+1}{2} + 1 \right] \\ &= \frac{n+3}{2n} \end{aligned}$$

Thus, if $n = 10$	$= \frac{(10)+3}{2(10)}$	if $n = 100$	$= \frac{(100)+3}{2(100)}$
	$= \frac{13}{20}$		$= \frac{103}{200}$

We can also find the limit of a sum as n approaches ∞ . [equal degree – ratio of leading coefficients]
Using the example above:

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i+1}{n^2} \\&= \lim_{n \rightarrow \infty} \frac{n+3}{2n} \\&= \frac{1}{2}\end{aligned}$$

ex. evaluate:

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\&= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i^2}{n^3} \\&= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^n i^2 \\&= \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \\&= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} \\&= \frac{2}{6} \\&= \frac{1}{3}\end{aligned}$$

Practice Questions

Day 1: Worksheet & pg. 260/ # 1 – 14

Day 2: Pg. 260/# 15 – 20, 23, 24, 27, 28, 29 – 33