APCalc Date:________________

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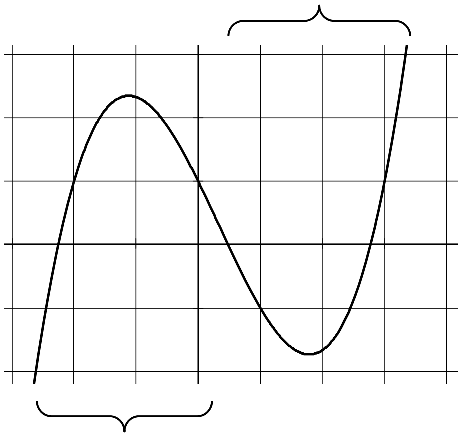
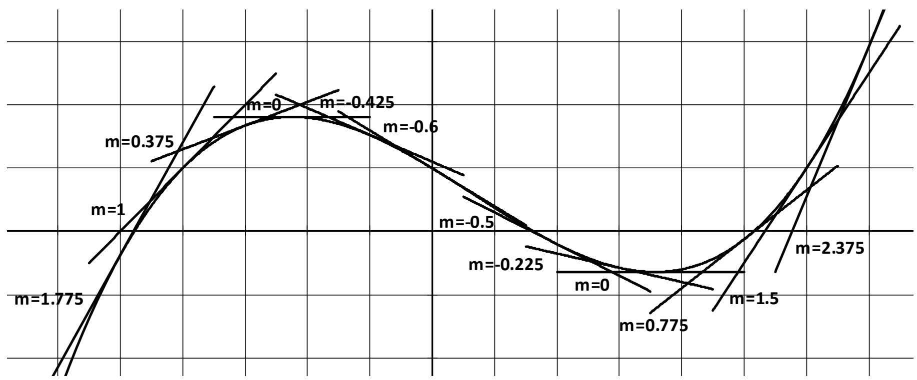
Unit 3: Applications of Differentiation
3.4 Concavity & The 2nd Derivative Test

Concavity – refers to whether the graph of a function is curving upward (concave up) or curving downward (concave down).

Concave up



Concave down



Tracking the values of various tangent slopes as x gets larger demonstrates that the slope values (ie. the derivative values) get smaller while the graph is concave down and get larger while the graph is concave up.

*applet

Definition of Concavity

Let $f(x)$ be a differentiable function on an interval I.

The graph of $f(x)$ is *concave upward* on I if $f'(x)$ is increasing on the interval.

The graph of $f(x)$ is *concave downward* on I if $f'(x)$ is decreasing on the interval.

From 3.3 we know that a function is increasing when its derivative is positive and is decreasing when its derivative is negative.

Thus, if $f'(x)$ is increasing then its derivative $f''(x) > 0$.

Similarly, if $f'(x)$ is decreasing then its derivative $f''(x) < 0$.

Test for Concavity

Let $f(x)$ be a twice differentiable function on an interval I.

If $f''(x) > 0$ for all x in I then the graph of $f(x)$ is concave upward on I.

If $f''(x) < 0$ for all x in I then the graph of $f(x)$ is concave downward on I.

ex. Determine the open intervals on which the graph of $f(x) = \frac{x^3}{3} - x$ is concave up or down.
*note – to solve this question we will determine the intervals on which $f'(x)$ is increasing or decreasing

Step 1: Identify the critical values of $f'(x)$, where $f''(x) = 0$ or $f''(x)$ is undefined.

$f'(x) = x^2 - 1$

*do not need criticals of f $(x = \pm 1)$ as these are possible extrema of f and have nothing to do with concavity

$f''(x) = 2x$ is always defined; critical at $x = 0$

Step 2: Set-up intervals and test values in $f''(x)$.

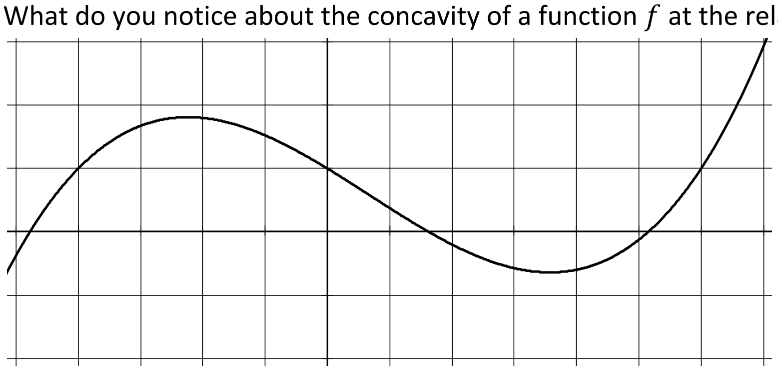
Interval	$(-\infty, 0)$	$(0, \infty)$
Test	$f''(-1) < 0$	$f''(1) > 0$
Conclusion	$f'(x)$ decreasing	$f'(x)$ increasing
	$\therefore f(x)$ concave down	$\therefore f(x)$ concave up

Concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.
*check with graphing calculator

The point where concavity changes is called a ***point of inflection***. In the previous example the point of inflection was $(0, 0)$.

Points of inflection possibly occur at criticals of $f'(x)$. [ie. where the derivative values reach a relative minimum or maximum]

What do you notice about the concavity of a function f at the relative extrema of f?



At a relative maximum the graph is concave downward.

$$f''(x) < 0$$

At a relative minimum the graph is concave upward.

$$f''(x) > 0$$

Thus, we can use the 1st derivative to identify the possible extrema locations of f [ie. the criticals of f] and then we can determine whether these are minimums or maximums by using the 2nd derivative to check the concavity at these possible extrema.

The 2^{nd} Derivative Test for Extrema

Let f be a function with a critical value c and a 2^{nd} derivative $f''(x)$ which exists on an interval containing c.

If $f''(c) > 0$ then $f(c)$ is a relative minimum.

If $f''(c) < 0$ then $f(c)$ is a relative maximum.

*If $f''(c) = 0$ then the test fails. Most likely this is because $f(c)$ is an inflection point, but we use the 1^{st} derivative test to check for any more extrema at $f(c)$.

ex. Use the 2^{nd} derivative test to identify the relative extrema of $f(x) = -3x^5 + 5x^3$

Step 1: Identify the critical values of f (where the possible extrema are).

$f'(x) = -15x^4 + 15x^2$ has no undefined values

$f'(x) = -15x^4 + 15x^2 = 0$

$-15x^2(x^2 - 1) = 0$

$x = 0$ or $x = \pm 1$

Step 2: Use the 2^{nd} derivative to determine concavity and conclude the nature of the extrema.

$f''(x) = -60x^3 + 30x$ *do NOT set equal 0, those are possible inflection points NOT extrema

$f''(-1) > 0$ Concave up $\therefore$ min	$f''(1) < 0$ Concave down $\therefore$ max	$f''(0) = 0$ test inconclusive – switch to 1^{st} derivative test $f'(-0.1) > 0$ $f'(0.1) > 0$ f is increasing before and after $x = 0$ $\therefore$ $(0, 0)$ is neither a min nor a max

Relative minimum at $(-1, -2)$ because $f'(-1) = 0$ and $f''(-1)$ is positive.
Relative maximum at $(1, 2)$ because $f'(1) = 0$ and $f''(1)$ is negative.

*check with graphing calculator

Practice Questions

Pg. 184/ # 1 – 19, 21 – 31, 35, 47 – 50, 58, 63, [77 – 80]

*bracketed questions are more challenging