APCalc Date:________________

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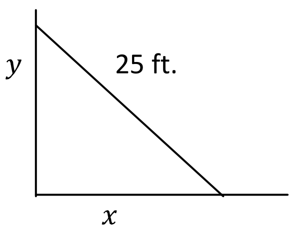
Unit 2: Differentiation
2.8 Related Rates

The rate of change of 2 or more variables with respect to time (t).

$\frac{dx}{dt}$ the rate of change of x with respect to t

$\frac{dy}{dt}$ the rate of change of y with respect to t

ex. A 25 ft. long ladder leans against a wall. The base of the ladder is pulled away from the wall at a rate of 3 ft/sec, how fast will the top of the ladder be sliding down the wall when the base is 15 ft. away from the wall?



$\frac{dx}{dt} = 3\ ft/sec$

find $\frac{dy}{dt}$ when $x = 15\ ft.$

*note: when $x = 15$

$y = \sqrt{25^2 - 15^2} = 20$

The lengths x and y are related by Pythagoras:

$$x^2 + y^2 = 25^2$$

Differentiate with respect to time:

$$\frac{d}{dt}[x^2] + \frac{d}{dt}[y^2] = \frac{d}{dt}[625]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y}$$

$$\frac{dy}{dt} = \frac{-(15)(3)}{(20)}$$

$$= \boxed{-\frac{9}{4} ft/\sec\ or\ -2.25\ ft/sec}$$

ex. If $5x + 3xy = 4$ and $\frac{dy}{dt} = -2$, determine $\frac{dx}{dt}$ when $x = 2$.

*when $x = 2$:

$$5(2) + 3(2)y = 4$$
$$6y = -6$$
$$y = -1$$

$$\frac{d}{dt}[5x] + \frac{d}{dt}[3xy] = \frac{d}{dt}[4]$$
$$5\frac{dx}{dt} + 3x\frac{dy}{dt} + 3y\frac{dx}{dt} = 0$$
$$5\frac{dx}{dt} + 3y\frac{dx}{dt} = -3x\frac{dy}{dt}$$
$$\frac{dx}{dt} = \frac{-3x\frac{dy}{dt}}{5+3y}$$
$$= \frac{-3(2)(-2)}{5+3(-1)}$$
$$= \frac{12}{2}$$
$$= \boxed{6}$$

ex. A spherical balloon is inflated with air at a rate of $5\pi\ cm^3/s$.

a) Determine the rate of change of the radius when $r = 5\ cm$

$\frac{dv}{dt} = 5\pi\ cm^3/s$; find $\frac{dr}{dt}$ when $r = 5\ cm$

$$V = {}^4/_3\,\pi r^3$$
$$\frac{dV}{dt} = \frac{d}{dt}\left[{}^4/_3\,\pi r^3\right]$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}$$
$$= \frac{5\pi}{4\pi(5)^2}$$
$$= \boxed{\frac{1}{20}\ cm/s}$$

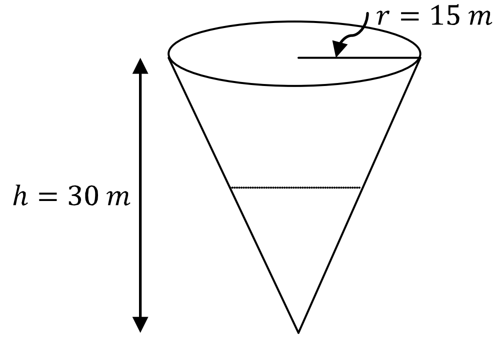
b) Determine the rate of change of the area of the cross-section when $r = 5\ cm$.

*note: because at the same instance as part a we can use $\frac{dr}{dt} = \frac{1}{20}$

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$
$$= 2\pi(5)\left(\frac{1}{20}\right)$$
$$= \boxed{\frac{\pi}{2}\ cm^2/s}$$

ex. A conical tank (vertex down) is being filled with water at a rate of $18\pi\ m^3/min$. If the tank has a height of 30 m and a radius of 15 m, how fast is the water level rising when the water is 12 m deep?

$\frac{dv}{dt} = 18\pi\ m^3/min$; find $\frac{dh}{dt}$ when $h = 12\ m$



$$V = \frac{1}{3}\pi r^2 h$$

There are too many variables in this formula, need to eliminate r:

in any cone, the ratio of radius to height is constant

ie.

$$\frac{r}{h} = \frac{15}{30} = \frac{1}{2}$$

$$r = \frac{h}{2}$$

$$\therefore\ V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{\pi}{12}h^3$$

$$\therefore\ \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

$$= \frac{4(18\pi)}{\pi(12)^2}$$

$$= \boxed{0.5\ m/min}$$

Practice Questions

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