APCalc Date:_____
Mr. C. Sdoutz

Unit 2: Differentiation
2.7 Implicit Differentiation

So far all functions we have differentiated have been expressed *explicitly* – y as a function of x.

ex. $y = \underbrace{3x^2 - 5x}$

explicit – only the input variable x (no y)

An equation given in terms of x and y is expressed *implicitly*.

ex. $xy = 1$

sometimes we can rewrite the relation explicitly and then differentiate as we have done previously.

$$y = \frac{1}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x^2}}$$

What if we cannot rewrite the relation explicitly?

ex. Given $x^2 - 2y^3 + 4y = 2$ determine the value of $\frac{dy}{dx}$.

differentiate all terms with respect to x

$$\frac{d}{dx}[x^2] - \frac{d}{dx}[2y^3] + \frac{d}{dx}[4y] = \frac{d}{dx}[2]$$

*note: treat y as an "inside" function of x and use the chain rule

$$2x - 6y^2\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

collect $\frac{dy}{dx}$ terms on one side

$$-6y^2\frac{dy}{dx} + 4\frac{dy}{dx} = -2x$$

factor $\frac{dy}{dx}$ & isolate

$$\frac{dy}{dx}(-6y^2 + 4) = -2x$$

$$\boxed{\frac{dy}{dx} = \frac{-2x}{-6y^2 + 4} = \frac{2x}{6y^2 - 4}}$$

ex. Determine the first derivative of $y^3 + 5y^2 - 3y - x^2 = -4$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[5y^2] - \frac{d}{dx}[3y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$3y^2\frac{dy}{dx} + 10y\frac{dy}{dx} - 3\frac{dy}{dx} - 2x = 0$$

$$3y^2\frac{dy}{dx} + 10y\frac{dy}{dx} - 3\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}(3y^2 + 10y - 3) = 2x$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{3y^2 + 10y - 3}}$$

ex. Determine the first derivative of $\sin(x + y) = 5xy^2$

$$\frac{d}{dx}\underbrace{[\sin(x+y)]}_{\text{Chain rule}} = \frac{d}{dx}\underbrace{[5xy^2]}_{\text{Product rule}}$$

$$cos(x+y)\frac{d}{dx}[x+y] = 5x\frac{d}{dx}[y^2] + y^2\frac{d}{dx}[5x]$$

$$\cos(x+y)\left(1 + \frac{dy}{dx}\right) = 5x\left(2y\frac{dy}{dx}\right) + y^2(5)$$

$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = 10xy\frac{dy}{dx} + 5y^2$$

$$\cos(x+y)\frac{dy}{dx} - 10xy\frac{dy}{dx} = 5y^2 - \cos(x+y)$$

$$\frac{dy}{dx}(\cos(x+y) - 10xy) = 5y^2 - \cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{5y^2 - \cos(x+y)}{\cos(x+y) - 10xy}}$$

ex. Find the 2nd derivative of $x^2 + y^2 = 25$

*find 1st derivative:

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[25]$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

*now differentiate each side again to find 2nd derivative (quotient law on right side)

$$\frac{d^2y}{dx^2} = \frac{y\frac{d}{dx}[-x] - (-x)\frac{d}{dx}[y]}{y^2}$$

$$= \frac{y(-1) + x\frac{dy}{dx}}{y^2}$$

$$= \frac{x\frac{dy}{dx} - y}{y^2}$$

*sub-in $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{x\left(\frac{-x}{y}\right) - y}{y^2}$$

$$= \frac{\frac{-x^2}{y} - y}{y^2}$$

$$= \frac{-x^2 - y^2}{y} \times \frac{1}{y^2}$$

$$= \frac{-(x^2 + y^2)}{y^3}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-25}{y^3}}$$

Practice Questions

Pg. 139/ # 1 – 7, 9 – 11, 15 – 23 odd, 35, 37, 39, 41