Unit 2: Differentiation
2.2 Differentiation & Continuity

Recall from 2.1 we constructed the formal definition of the derivative of $f(x)$ as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

ex. If $f(x) = x^2$ determine the value of $f'(3)$.

1st determine $f'(x)$:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= 2x + (0) \\ f'(x) &= 2x \end{aligned}$$

Then sub in the given value of x:

$$f'(3) = 2(3) = \boxed{6}$$

There is also an alternate form of the derivative which bypasses the $f'(x)$ function and gives you the **value** of the derivative $f'(c)$:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In the previous example:

$$\begin{aligned} f'(3) &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} \\ &= \lim_{x \to 3} x + 3 \\ &= (3) + 3 = \boxed{6} \end{aligned}$$

Either way we can say that $f(x) = x^2$ is differentiable at $x = 3$ since the derivative $f'(3)$ exists.

We use the alternate form of the derivative to investigate the relation between differentiability and continuity.

From unit 1 (Limits) we know that

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

will exist only if the one sided-limits exist and equal each other.
ie.

$$\underbrace{\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}}_{\text{derivative from the left}}=\underbrace{\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}}_{\text{derivative from the right}}$$

Since the derivative can be interpreted as the slope of the tangent line to the curve we can conclude that if a function has a discontinuity at $x=c$ (hole, jump, asymptote) then the function will NOT be differentiable at $x=c$. (How can you have a tangent line at a discontinuity?)

So a discontinuity will automatically mean not differentiable.

But, will continuity automatically imply differentiability?

Will look at two special cases:

<u>A graph with an abrupt turn.</u>

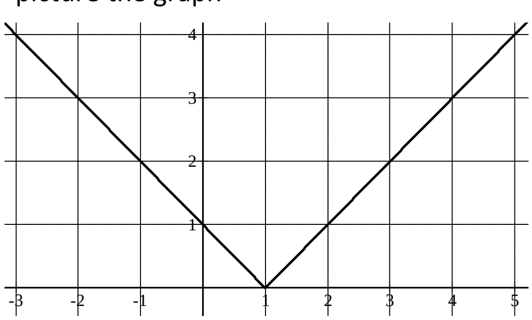
ex. Is $f(x)=|x-1|$ differentiable at $x=1$?

check differentiability from each side:

$$\begin{aligned}\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}&=\lim_{x\to 1^-}\frac{|x-1|-0}{x-1}\\&=\lim_{x\to 1^-}\frac{-(x-1)}{x-1}\\&=\lim_{x\to 1^-}-1\\&=-1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}&=\lim_{x\to 1^+}\frac{|x-1|-0}{x-1}\\&=\lim_{x\to 1^-}\frac{x-1}{x-1}\\&=\lim_{x\to 1^-}1\\&=1\end{aligned}$$

The two one-sided derivatives are not equal $\therefore$ $f'(1)$ does not exist. The function is not differentiable at $x=1$ even though it is everywhere continuous.
*picture the graph



Visually we see exactly what the derivatives told us above. As you approach $x=1$ from the left the slope of the function (and thus the derivative) is equal to -1, but as you approach $x=1$ from the right the slope of the function (and thus the derivative) is equal to +1. This abrupt change in derivative values is caused by the abrupt change in the graph of the function.

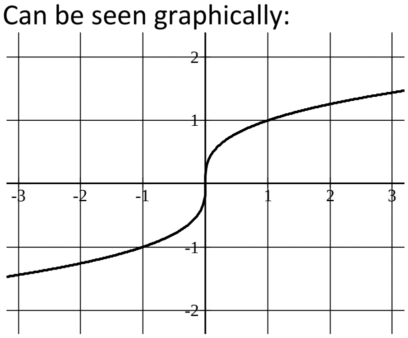
A graph with a vertical tangent line.

ex. Is $f(x) = \sqrt[3]{x}$ differentiable at $x = 0$?

$f(x)$ is everywhere continuous but:

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{\sqrt[3]{x} - 0}{x - 0} \\ &= \lim_{x\to 0} \frac{x^{1/3}}{x} \\ &= \lim_{x\to 0} x^{-2/3} \\ &= \lim_{x\to 0} \frac{1}{x^{2/3}} \\ &= \infty \end{aligned}$$

The derivative $f'(0)$ does not exist because there is a vertical tangent line at $x = 0$, and thus an undefined slope (derivative) value.
Can be seen graphically:



Thus we have shown that continuity does NOT guarantee differentiability.

The reverse can be stated though:

Theorem
If a function is differentiable at $x = c$ then the function is continuous at $x = c$

Summary:
1) Discontinuity = NOT differentiable
2) Differentiability = continuity
3) Continuity $\neq$ differentiability

Practice Questions

Pg. 100/ # 45, 47, 49, 51 – 61, 63, 65, 69 – 71