APCalc Date:________________

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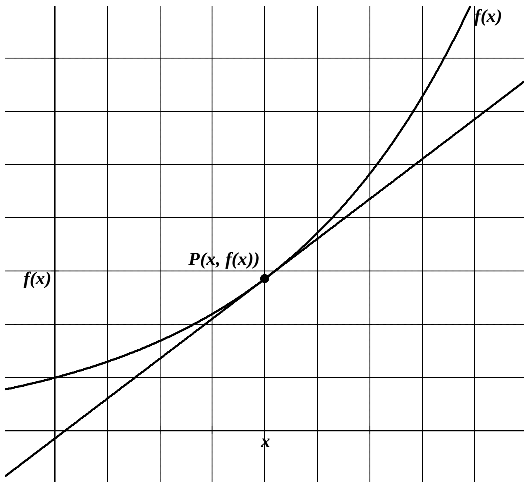
Unit 2: Differentiation

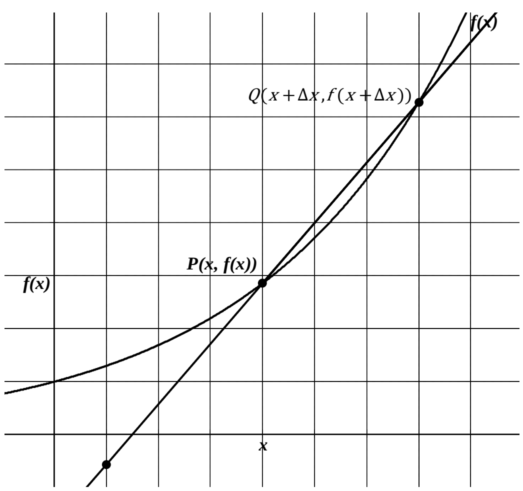
2.1 The Derivative

Tangent line problem – How do we find the slope of a tangent line at a point (the straight line that "just touches" the curve at that point)?

ex. Given the function $f(x)$ shown, find the slope of the tangent line at point $P\big(x, f(x)\big)$.



Use a secant line (a line that crosses through exactly 2 points) that passes through $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$:



Slope of the secant line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x}$$

$$= \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If we slide the point Q closer and closer to P (ie. as Δx approaches 0) then the slope of the secant line will approach the slope of the tangent line. [online applet]

ie. the slope of the tangent line:

$$\boxed{m_{tangent} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

ex. Find the slope of the tangent to $f(x) = 3x^2 + 2$ at $x = 1$.

$$\begin{aligned} m_{tangent} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[3(x+\Delta x)^2 + 2] - [3x^2 + 2]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3(x^2 + 2x\Delta x + \Delta x^2) + 2 - 3x^2 - 2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3x^2 + 6x\Delta x + 3\Delta x^2 + 2 - 3x^2 - 2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3\Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 6x + 3\Delta x \\ &= 6x + 3(0)^2 \\ m_{tangent} &= 6x \end{aligned}$$

*note: the limit is a function which will tell us the slope of the tangent at **<u>any</u>** point, we just need to sub in the appropriate value of x

$\therefore$ at $x = 1$ the tangent slope $= 6(1) = \boxed{6}$

For the same function write the equation of the tangent line at $x = -2$.

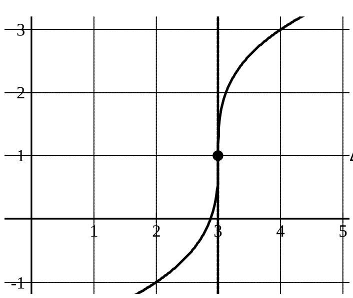
at $x = -2$ tangent slope $= 6(-2) = -12$
the point is $\left(-2, f(-2)\right) = (-2, 3(-2)^2 + 2) = (-2, 14)$
use <u>point-slope form</u> to write the equation of the line:

$y - y_1 = m(x - x_1)$

$y - 14 = -12(x - (-2))$

$$\boxed{y = -12x - 10 \quad \text{or} \quad 12x + y + 10 = 0}$$

There is a possibility of a vertical tangent line with undefined slope:



In this example at $x = 3$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \pm\infty$$

ie. the limit is undefined at this particular value of x

The limit used to define the slope of the tangent line is also used to define one of the two fundamental operations of Calculus – ***differentiation***.

Definition of the derivative:

The derivative of a function f at a value x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

"f prime of x" – denotes the derivative

*$f'(x)$ is the slope of the tangent line, it is the instantaneous rate of change of the function f

The process of finding the *derivative* is called *differentiation*. If the derivative exists at x then the function is *differentiable* at x.

*other notation to represent the derivative:

$$f'(x) = y' = \underbrace{\frac{dy}{dx} = \frac{d[f(x)]}{dx}}$$

"y prime" "the derivative of y (or $f(x)$) with respect to x"

Leibniz Notation

ex. Given $f(x) = 2x^2 + x$ determine $f'(1)$ and the equation of the tangent line at $x = 1$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x)] - [2x^2 + x]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2(x^2 + 2x\Delta x + \Delta x^2) + x + \Delta x - 2x^2 - x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x^2 + 4x\Delta x + 2\Delta x^2 + x + \Delta x - 2x^2 - x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2\Delta x^2 + \Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 4x + 2\Delta x + 1 \\ &= 4x + 2(0) + 1 \\ f'(x) &= 4x + 1 \end{aligned}$$

$$\therefore f'(1) = 4(1) + 1 = \boxed{5}$$

Thus, slope of the tangent line at $(1, 3)$ is 5.

The tangent equation is: $y - 3 = 5(x - 1)$

$$\boxed{y = 5x - 2 \text{ or } 5x - y - 2 = 0}$$

You can calculate (**check!)** derivative values at specific values of x using your calculator (but it cannot calculate $f'(x)$ functions).

math → nDeriv (#8)

nDeriv(function, variable, value)

In the previous example: $f(x) = 2x^2 + x$ determine $f'(1)$

$f'(1) = nDeriv(2x^2 + x, x, 1) = 5$

Practice Questions

Pg. 99/ # 6, 7, 9, 11, 14, 15, 17, 19, 21, 23, 27 - 32