APCalc
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Date:________________

Unit 1: Limits
1.7 Limits at Infinity

Examine the "end behaviour" of a function as x approaches $\pm\infty$.

ie. $\lim_{x\to\pm\infty} f(x)$

ex. $f(x) = \frac{4x^2}{x^2+2}$
investigate the end behaviour using a table:

x approaches $-\infty$ $\longleftarrow$ $\longrightarrow$ x approaches ∞

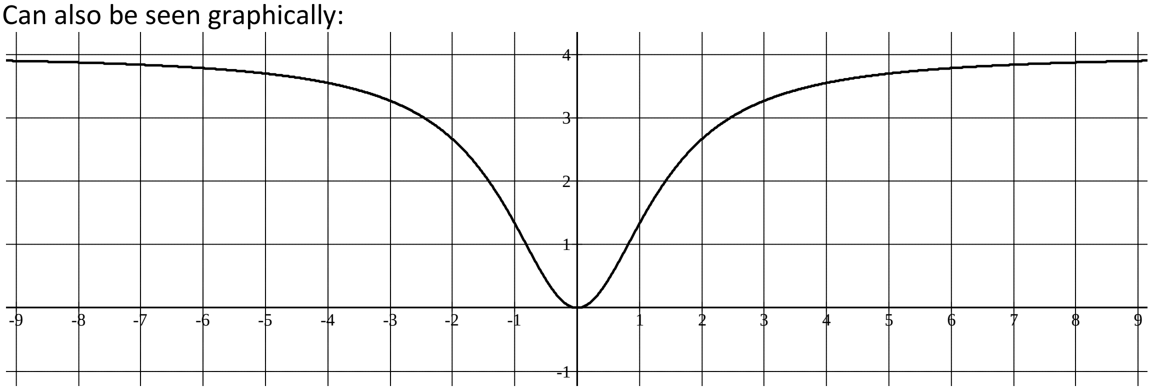
x	-100	-10	-1	0	1	10	100
$f(x)$	3.9992	3.9216	1.333	0	1.333	3.9216	3.9992

$f(x)$ approaches 4 $\qquad$ $f(x)$ approaches 4

$$\lim_{x\to-\infty} f(x) = 4 \text{ and } \lim_{x\to\infty} f(x) = 4$$

$\therefore$ There is a *horizontal asymptote* at $y = 4$.

Can also be seen graphically:



The line $y = L$ is a horizontal asymptote of f if and only if $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$.

*note: a function can have at most two horizontal asymptotes

Theorem (Limits at Infinity)
If r is a positive rational number and c is any real number, then $\lim_{x\to\pm\infty} \frac{c}{x^r} = 0$

ie. a constant divided by a value approaching infinity will approach zero

ex. $\lim_{x\to\infty} \left(5 - \frac{2}{x^2}\right) = 5 - 0 = \boxed{5}$

To evaluate a limit at ∞ infinity divide all terms by the <u>largest</u> power of x in the <u>denominator</u>.
ex.

$$\lim_{x\to\infty}\frac{4x^3}{x^3+2}$$

*divide all terms by x^3

$$=\lim_{x\to\infty}\frac{\frac{4x^3}{x^3}}{\frac{x^3}{x^3}+\frac{2}{x^3}}$$

$$=\lim_{x\to\infty}\frac{4}{1+\frac{2}{x^3}}$$

$$=\frac{4}{1+0}$$

$$\boxed{=4}$$

*if degree of the numerator = degree of the denominator then the limit at infinity is the ratio of the leading coefficients (horizontal asymptote x =limit)

ex.

$$\lim_{x\to\infty}\frac{2x^2+5}{3x^4+1}$$

*divide all terms by x^4

$$=\lim_{x\to\infty}\frac{\frac{2x^2}{x^4}+\frac{5}{x^4}}{\frac{3x^4}{x^4}+\frac{1}{x^4}}$$

$$=\lim_{x\to\infty}\frac{\frac{2}{x^2}+\frac{5}{x^4}}{3+\frac{1}{x^4}}$$

$$=\frac{0+0}{3+0}$$

$$\boxed{=0}$$

*if degree of the numerator < degree of the denominator then the limit at infinity is 0 (x-axis horizontal asymptote)

ex.

$$\lim_{x\to\infty}\frac{5x^3+2x}{7x^2-3}$$

*divide all terms by x^2

$$=\lim_{x\to\infty}\frac{\frac{5x^3}{x^2}+\frac{2x}{x^2}}{\frac{7x^2}{x^2}-\frac{3}{x^2}}$$

$$=\lim_{x\to\infty}\frac{5x+\frac{2}{x}}{7-\frac{3}{x^2}}$$

$$=\frac{5(\infty)+0}{7-0}$$

$=\infty$ ie. the limit DNE

*if degree of the numerator > degree of the denominator then the limit at infinity DNE (no horizontal asymptote) ex.

$$\lim_{x\to\infty}\frac{7x^5-3x^2+2}{1-2x^5}=\boxed{-\frac{7}{2}}$$

Rational functions will have the same asymptote as x approaches $-\infty$ or ∞.

Radical functions can have different horizontal asymptotes as x approaches $-\infty$ or ∞.
ex.

$$\lim_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}}$$

$x\to\infty \quad \therefore x>0$
then $x=\sqrt{x^2}$

$$=\lim_{x\to\infty}\frac{\frac{3x}{\sqrt{x^2}}-\frac{2}{\sqrt{x^2}}}{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}$$

$$=\lim_{x\to\infty}\frac{\frac{3x}{x}-\frac{2}{x}}{\sqrt{\frac{2x^2+1}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}$$

$$=\frac{3-0}{\sqrt{2+0}}$$

$$=\frac{3}{\sqrt{2}}$$

$$\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}}$$

$x\to\infty \quad \therefore x<0$
then $x=-\sqrt{x^2}$

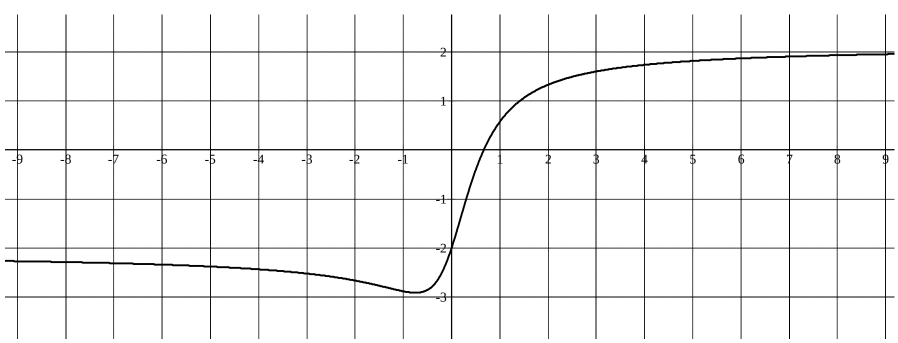
$$=\lim_{x\to\infty}\frac{\frac{3x}{-\sqrt{x^2}}-\frac{2}{-\sqrt{x^2}}}{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}}$$

$$=\lim_{x\to\infty}\frac{\frac{3x}{x}-\frac{2}{x}}{-\sqrt{\frac{2x^2+1}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}$$

$$=\frac{3-0}{-\sqrt{2+0}}$$

$$=-\frac{3}{\sqrt{2}}$$



*note: it is OK that the graph crosses the horizontal asymptotes, the asymptotic boundary behaviour does not occur until the extreme left and right of the graph (ie. as x approaches $\pm\infty$)

Practice Questions

Pg. 193 / # 1 – 4, 11 – 19, 27, $\underbrace{35-42,\ 52}_{\text{*find horizontal asymptotes}}$