APCalc   Date:________________

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Unit 1: Limits
1.4 Intervals and Continuity I

Interval Notation – alternative to inequality statements

ex.

*exclusive interval from a to b (not including a and b)

$a < x < b$

- an *open* interval

Interval notation: (a, b)

*looks like a coordinate but from the context of the situation you will know that it is an interval

ex.

*inclusive interval from a to b (including a and b)

$a \leq x \leq b$

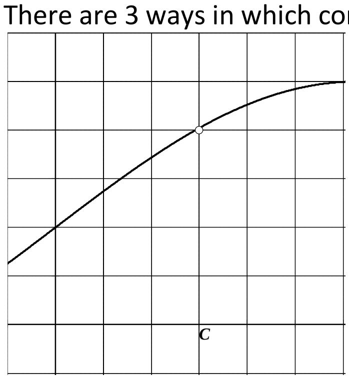
- a *closed* interval

Interval notation: $[a, b]$

Can also have *mixed* intervals: $(1,5]$ is the interval from, but not including, 1 to, and including, 5

Continuity – informally, a function is continuous at a point if there are no "interruptions" at the point (ie. no holes, jumps, or asymptotes)

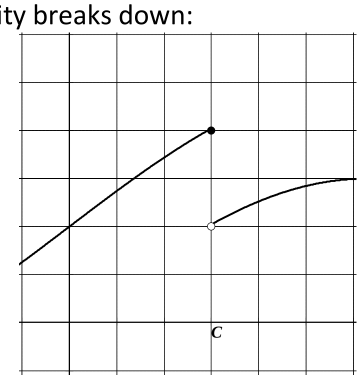
There are 3 ways in which continuity breaks down:



not continuous at c

(removable)

$f(c) = \infty$ (undefined)

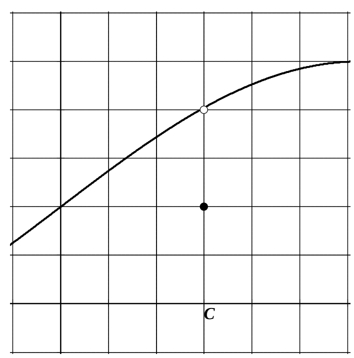


not continuous at c

(non-removable)

$\lim_{x \to c} f(x) = DNE$

*can also be caused by asymptotes



not continuous at c

(removable)

$\lim_{x \to c} f(x) \neq f(c)$

Definition of Continuity at a Point

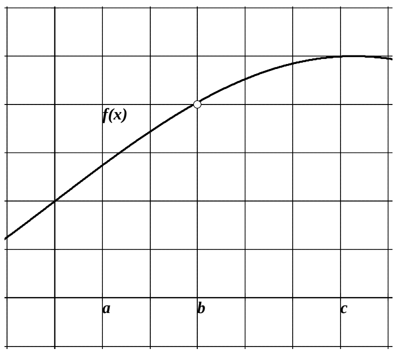
A function f is continuous at c if and only if the following **three** conditions are met:

1) $f(c)$ is defined
2) $\lim_{x \to c} f(x)$ exists
3) $\lim_{x \to c} f(x) = f(c)$

Definition of Continuity on an Open Interval

A function f is continuous on (a, b) if it is continuous at each point on the interval.

* can draw the graph between a and b without lifting pencil off the page



$f(x)$ is continuous on intervals:

$(-\infty, a), (-\infty, b), (a, b), (b, c), (c, \infty)$

$f(x)$ is <u>not</u> continuous on intervals:

$(-\infty, c), (a, c), (a, \infty), (-\infty, \infty)$

If f is continuous on $(-\infty, \infty)$ then it is called <u>*everywhere continuous*</u>

*<u>**ALL**</u> polynomial functions are <u>*everywhere continuous*</u>.

When a function f is not continuous at c then we say f has a discontinuity at c.

ex.

$$f(x) = \frac{x^2 + 2x}{x}$$

has a discontinuity at $x = 0$ since $f(0)$ is undefined (but is continuous everywhere else)

Discontinuities are classified as <u>*removable*</u> or <u>*non-removable*</u>. Non-removable discontinuities are further classified as <u>*jump*</u> or <u>*infinite*</u> (asymptote).

A removable discontinuity is a hole in the graph (an undefined value that is not an asymptote).

A discontinuity is removable at $x = c$ if $f(c) = \frac{0}{0}$.

ex.

$$f(x) = \frac{x^2 + 2x}{x}$$

has a removable discontinuity at $x = 0$ since $f(0) = \frac{0}{0}$ (ie. there is a hole in the graph at $x = 0$)

We can "remove" the discontinuity by redefining $f(x)$ as a piecewise function and assigning a value to $f(0)$ (ie. we "fill in" the hole). To do this we need to know the exact location of the hole $(0, y)$. To do this we find the limit as x approaches 0.

$$\lim_{x \to 0} \frac{x^2 + 2x}{x}$$
$$= \lim_{x \to 0} \frac{x(x + 2)}{x}$$
$$= \lim_{x \to 0} x + 2$$
$$= (0) + 2 = 2$$ Thus the hole is at (0, 2).

$$\therefore f(x) = \begin{cases} \frac{x^2+2x}{x}; x \neq 0 \\ 2 \,; x = 0 \end{cases}$$ is everywhere continuous

ex. Discuss the continuity of $f(x) = \begin{cases} x+1, & x<2 \\ 2x+1, & x \geq 2 \end{cases}$

The polynomial parts of the function $x+1$ and $2x+1$ are continuous everywhere so the only value that needs analysis is $x=2$ (the transition point between the pieces).

Use the 3 part definition of continuity at a point to determine continuity at $x=2$:

1) $f(c)$ is defined

$$f(2) = 2(2) + 1 = 5 \quad \text{yes}$$

2) $\lim_{x \to c} f(x)$ exists

$$\lim_{x \to 2^-} f(x) = (2) + 1 = 3 \qquad \lim_{x \to 2^+} f(x) = 2(2) + 1 = 5$$

$$\therefore \lim_{x \to 2} f(x) = DNE \quad \text{no}$$

3) $\lim_{x \to c} f(x) = f(c)$ no

Since the limit did not exist we know that $x=2$ is a discontinuity. More importantly, the one-sided limits being different means it is a non-removable jump discontinuity.

ex. Discuss the continuity of $f(x) = \begin{cases} x^3+2, & x \leq 1 \\ 2x+1, & x > 1 \end{cases}$

The polynomial parts of the function x^3+2 and $2x+1$ are continuous everywhere so the only value that needs analysis is $x=1$ (the transition point between the pieces).

Use the 3 part definition of continuity at a point to determine continuity at $x=1$:

1) $f(c)$ is defined

$$f(1) = (1)^3 + 2 = 3 \quad \text{yes}$$

2) $\lim_{x \to c} f(x)$ exists

$$\lim_{x \to 1^-} f(x) = (1)^3 + 2 = 3 \qquad \lim_{x \to 1^+} f(x) = 2(1) + 1 = 3$$

$$\therefore \lim_{x \to 1} f(x) = 3 \quad \text{yes}$$

3) $\lim_{x \to c} f(x) = f(c)$

$$\lim_{x \to 1} f(x) = 3 = f(1) \quad \text{yes}$$

f is continuous at $x=1$ and thus f is everywhere continuous.

A function may have multiple discontinuities and they might be different types.

ex.

$$f(x) = \frac{x+2}{x^2-4}$$

has two discontinuities at $x = \pm 2$ since $f(\pm 2) = \infty$

$f(2) = \frac{4}{0}$ and $f(-2) = \frac{0}{0}$

Thus, $x = 2$ is a non-removable infinite asymptote while $x = -2$ is a removable discontinuity.

$$* \lim_{x\to\pm 2} \frac{x+2}{x^2-4} = \lim_{x\to\pm 2} \frac{x+2}{(x+2)(x-2)} = \lim_{x\to\pm 2} \frac{1}{x-2} = \begin{cases} \lim_{x\to -2} \frac{1}{x-2} = \frac{1}{(-2)-2} = -\frac{1}{4} \\ \lim_{x\to 2} \frac{1}{x-2} = \frac{1}{(2)-2} = \infty \end{cases}$$

ie. hole at $\left(-2, -\frac{1}{4}\right)$, asymptote $x = 2$

Practice Questions

Pg. 75/ # 1, 3, 27 – 30, 31 – 46 (if removable redefine the function to make it everywhere continuous)