Unit 1: Limits
1.3 Limit Properties, Special Trig Limits, & One-sided Limits

Limit properties

Let b and c be real numbers and n be a positive rational. Let f and g be functions such that

$$\lim_{x\to c} f(x) = L \ \& \ \lim_{x\to c} g(x) = K$$

1) Scalar Multiple

$$\lim_{x\to c}(bf(x)) = b\left(\lim_{x\to c} f(x)\right) = bL$$

ex. if $\lim_{x\to 5} f(x) = 2$

then $\lim_{x\to 5} 3f(x) = 3(2) = 6$

2) Sum or Difference

$$\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm K$$

ex. if $\lim_{x\to 0} f(x) = -2$ and $\lim_{x\to 0} g(x) = 3$

then $\lim_{x\to 0}(g(x) - f(x)) = 3 - (-2) = 5$

3) Product

$$\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \lim_{x\to c} g(x) = LK$$

ex. if $\lim_{x\to 0} f(x) = -2$ and $\lim_{x\to 0} g(x) = 3$

then $\lim_{x\to 0}(f(x)g(x)) = 3(-2) = -6$

4) Quotient

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{K} \ ; k \neq 0$$

ex. if $\lim_{x\to 0} f(x) = -2$ and $\lim_{x\to 0} g(x) = 3$

then $\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right) = \frac{3}{-2} = -\frac{2}{3}$

5) Power

$$\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n = L^n$$

ex. if $\lim_{x\to 5} f(x) = 2$

then $\lim_{x\to 5}(f(x))^3 = (2)^3 = 8$

6) Composite

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(K)$$

ex. if $f(x) = \sqrt{x}$ & $g(x) = x^2 + 5$

then $\lim_{x \to 2} f\big(g(x)\big) = f(\lim_{x \to 2} g(x))$

$= f(\lim_{x \to 2} x^2 + 5)$

$= f((2)^2 + 5)$

$= f(9)$

$= \sqrt{9}$

$= \boxed{3}$

<u>Two Special Trig Limits</u>

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad\qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0}$$

ex.

$$\lim_{x \to 0} \frac{\sin x \cos x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} \times \lim_{x \to 0} \frac{\sin x}{x} = \frac{\cos(0)}{6} \times 1 = \boxed{\frac{1}{6}}$$

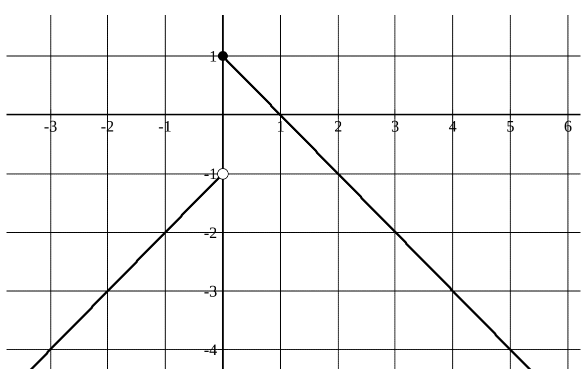
$$\lim_{x \to 0} \frac{1 + 3x - \cos x}{x} = \lim_{x \to 0} \frac{3x}{x} + \lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} 3 + 0 = \boxed{3}$$

$$\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{x \to 0} \frac{5 \sin 5x}{5x} = 5 \lim_{x \to 0} \frac{\sin 5x}{5x} = 5(1) = \boxed{5}$$

<u>One-Sided Limits</u>

Instead of analyzing the limit of a function as a value of x is approached from <u>both</u> sides we can analyze the behaviour of a function as it approaches x from one side.

ex. $f(x) = \begin{cases} x - 1; x < 0 \\ -x + 1; x \geq 0 \end{cases}$



As x approaches 0 from the left $f(x)$ approaches -1.

This is called the limit from the left and is denoted

$$\lim_{x \to 0^-} f(x) = -1$$

As x approaches 0 from the right $f(x)$ approaches 1.

This is called the limit from the right and is denoted

$$\lim_{x \to 0^+} f(x) = 1$$

The overall limit, however, does not exist (DNE) because the two one-sided limits do not equal each other. By stating the one-sided limits we can visualize (without needing a graph) this jump in the function (called a <u>jump discontinuity</u>).

Theorem – Existence of a Limit

Let f be a function and let c & L be real numbers. The limit of $f(x)$ as x approaches c is equal to L if and only if both one-sided limits of $f(x)$ as x approaches c are equal to L.

$$\text{ie. } \lim_{x\to c} f(x) = L \text{ iff } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

For non-piecewise function (where jumps like the one in the previous example are not possible) we can evaluate one-sided limits using our analytical techniques from lesson 1.2 (substitution, cancellation, rationalization) as we know the two one-sided limits will equal each other and equal the overall limit value.

ex.

$$\lim_{x\to 2^+} \frac{x^2}{x+2} = \frac{(2)^2}{(2)+2} = \boxed{1}$$

One-sided limits are more interesting for piecewise functions.

ex.

$$\text{if } f(x) = \begin{cases} 2x+3; x<1 \\ x^2+4; x>1 \end{cases} \quad \text{determine } \lim_{x\to 1} f(x)$$

*note: any other limit could be found by substitution in the appropriate piece of the function

Evaluate the two one-sided limits:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x+3 = 2(1)+3 = 5$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2+4 = (1)^2+4 = 5$$

$$\therefore \lim_{x\to 1} f(x) = \boxed{5}$$

From this analysis we can visualize that there is a hole in the graph of $f(x)$ at $x=1$ since $f(1)$ is undefined but the limit exists (this is called a <u>removable discontinuity</u>)

ex.

$$\text{determine } \lim_{x\to 3} \frac{|x-3|}{x-3}$$

The function $f(x) = \frac{|x-3|}{x-3}$ can be redefined as a piecewise function.

$\frac{|x-3|}{x-3} = \frac{-(x-3)}{x-3} = -1 \text{ if } x<3$

$\frac{|x-3|}{x-3} = \frac{x-3}{x-3} = 1 \text{ if } x \geq 3$

$$\therefore f(x) = \begin{cases} -1;\ x<3 \\ 1; x\geq 3 \end{cases}$$

$$\therefore \lim_{x\to 3^-} f(x) = -1 \quad \& \quad \lim_{x\to 3^+} f(x) = 1$$

$$\therefore \lim_{x\to 3} \frac{|x-3|}{x-3} = DNE$$

This analysis suggests a jump discontinuity at $x=3$.

Practice Questions

Pg. 64/ # 29 – 32, 57 – 66, 69

Pg. 76/ # 5 – 22